

MODULAR CURVATURE AND MORITA EQUIVALENCE

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ABSTRACT. The curvature of the noncommutative torus T_θ^2 ($\theta \in \mathbb{R} \setminus \mathbb{Q}$) endowed with a noncommutative conformal metric has been the focus of attention of several recent works. Continuing the approach taken in the paper [A. Connes and H. Moscovici, Modular curvature for noncommutative two-tori, J. Amer. Math. Soc. 27 (2014), 639–684] we extend the study of the curvature to twisted Dirac spectral triples constructed out of Heisenberg bimodules that implement the Morita equivalence of the C^* -algebra $A_\theta = C(T_\theta^2)$ with other toric algebras $A_{\theta'} = C(T_{\theta'}^2)$. In the enlarged context the conformal metric on T_θ^2 is exchanged with an arbitrary Hermitian metric on the Heisenberg $(A_\theta, A_{\theta'})$ -bimodule E' for which $\text{End}_{A_{\theta'}}(E') = A_\theta$. We prove that the Ray-Singer log-determinant of the corresponding Laplacian, viewed as a functional on the space of all Hermitian metrics on E' , attains its extremum at the unique Hermitian metric whose corresponding connection has constant curvature. The gradient of the log-determinant functional gives rise to a noncommutative analogue of the Gaussian curvature. The genuinely new outcome of this paper is that the latter is shown to be independent of any Heisenberg bimodule E' such that $A_\theta = \text{End}_{A_{\theta'}}(E')$, and in this sense it is Morita invariant. To prove the above results we extend Connes' pseudodifferential calculus to Heisenberg modules. The twisted version, which offers more flexibility even in the case of trivial coefficients, could potentially be applied to other problems in the elliptic theory on noncommutative tori. A noteworthy technical feature is that we systematize the computation of the resolvent expansion for elliptic differential operators on noncommutative tori to an extent which makes the (previously employed) computer assistance unnecessary.

INTRODUCTION

The concept of intrinsic curvature, which lies at the very core of geometry, has only recently begun to be comprehended in the noncommutative framework. As its earliest form, the Gaussian curvature, arose for Riemann surfaces, it was natural to look first at the noncommutative 2-torus T_θ^2 ($\theta \in \mathbb{R} \setminus \mathbb{Q}$), the simplest but nevertheless revealing example of a noncommutative surface. Tools for attacking this problem were developed early on, in Connes' seminal *Comptes Rendus* note [CON80]. They were applied in [CoCo93, CoTr11] to the computation of the value at 0 of the zeta function of the Laplacian associated to a translation-invariant

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conformal metric on T_θ^2 , or equivalently to the computation of the total curvature for such a metric, verifying the validity of the Gauss-Bonnet formula. The calculation of the full, not just the total, curvature was completed in [CoMo14] and also in [FAKH13], with the partial aid of different computer algebra systems. The resulting formula involves two kinds of functions of the modular operator associated to the conformal factor. One of them is the Bernoulli generating function in the modular operator, applied to the Laplacian of the conformal factor. The other term was given in [CoMo14] a conceptual explanation, as a consequence of expressing the curvature as the gradient of the Ray-Singer log-determinant of the varying Laplacian. It was also shown in [CoMo14] that the Ray-Singer functional attains its extreme value only at the flat metric.

In this paper we extend the study of the curvature to twisted Dirac spectral triples constructed out of Heisenberg bimodules that implement the Morita equivalence of the C^* -algebra $A_\theta = C(T_\theta^2)$ with other toric algebras. The enlarged context sets the scene for exploiting the Morita equivalence, which is shown to play a triple role. First of all, it exchanges the special (conformal) metric on the base with a completely general metric on the bundle. Secondly, it confers to the Ray-Singer functional a status akin to the Connes-Rieffel noncommutative Yang-Mills functional. Thirdly, and most surprisingly, it leads to a noncommutative analogue of the Gaussian curvature which is Morita invariant.

To convey the flavor of our main results in an informal yet suggestive manner, we shall appeal to the analogy between the basic Heisenberg modules over $A_\theta = C(T_\theta^2)$ and the Spin^c structures of an elliptic curve. The Heisenberg equivalence bimodules are finitely generated projective modules that implement the Morita equivalences between T_θ^2 and other tori $T_{\theta'}^2$, with θ' necessarily in the orbit of θ under the action of $\text{PSL}(2, \mathbb{Z})$ on the real projective line (cf. [CON80, RIE81]). Introduced in [CON80], these modules have an attractive geometric underpinning which we quickly recall (cf. [CON82]). The free homotopy classes of closed geodesics on the flat torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ are parametrized by the rational projective line $\mathbb{P}^1(\mathbb{Q}) \equiv \mathbb{Q} \cup \{\frac{1}{0}\}$. A pair of relatively prime integers $(d, c) \in \mathbb{Z}^2$ determines a family of lines of slope $\frac{d}{c}$, which project onto simple closed geodesics in the same free homotopy class. Let $N_{c,d}$ denote the primitive closed geodesic of slope $\frac{d}{c}$ passing through the base point of T^2 . Consider now the Kronecker foliation \mathcal{F}_θ of irrational slope $\theta \in (0, 1)$. Each geodesic $N_{c,d}$ gives a complete transversal to the foliation \mathcal{F}_θ , equipped with a holonomy pseudogroup. Choosing $a, b \in \mathbb{Z}$ such that $ad - bc = 1$, the convolution algebra of the corresponding étale holonomy groupoid can be identified with the crossed product algebra $C(\mathbb{R}/\mathbb{Z}) \rtimes_{\theta'} \mathbb{Z}$, where $1 \in \mathbb{Z}$ acts by the rotation of angle $\theta' = \frac{a\theta+b}{c\theta+d}$, which is isomorphic to $A_{\theta'}$; in particular, A_θ corresponds to $N_{0,1}$. The holonomy groupoids associated to the transversals $N_{c,d}$ and $N_{0,1}$ are Morita equivalent, and the $(A_{\theta'}, A_\theta)$ -bimodule $E(g, \theta)$ implementing the Morita equivalence between their C^* -algebras has a compelling geometric description ([CON80], also §1.2.1 below); in particular its smooth version $\mathcal{E}(g, \theta)$ carries a canonical Hermitian connection of constant curvature.

Fixing a complex structure on T^2 with modular parameter $\tau \in \mathbb{C}$, $\Im \tau > 0$, gives rise to a spectral triple over A_θ with operator D isospectral to the operator $\partial_\tau + \partial_\tau^*$ on T^2 . We regard D as the analogue of the Dirac operator associated

to a fundamental Spin^c -structure, and the collection of Heisenberg bimodules $\text{Mor}(T_\theta^2) := \{E(g, \theta); g \in \text{PSL}(2, \mathbb{Z})\}$ as the counterpart of the set of Spin^c -structures over an elliptic curve. For each $\mathcal{E} = \mathcal{E}(g, \theta)$ one forms the spectral triple with twisted Dirac operator $D_{\mathcal{E}} = \partial_{\mathcal{E}} + \partial_{\mathcal{E}}^*$ with coefficients in \mathcal{E} . It has a natural transposed with coefficients in the $(\mathcal{A}_\theta, \mathcal{A}_{\theta'})$ -bimodule $\mathcal{E}' = \mathcal{E}(g^{-1}, g \cdot \theta)$; here \mathcal{A}_θ stands for the usual smooth subalgebra of A_θ . By analogy with Connes' spectral characterization of Spin^c -manifolds [Con13], these are precisely the spectral triples which confer to T_θ^2 the structure of a noncommutative manifold endowed with a metric structure.

Since \mathcal{A}_θ coincides with the endomorphism algebra $\text{End}_{\mathcal{A}_{\theta'}}(\mathcal{E}')$, an arbitrary change of Hermitian structure on the $\mathcal{A}_{\theta'}$ -module \mathcal{E}' amounts to the choice of an invertible positive element $k \in \mathcal{A}_\theta$. Our first main result (Theorem 1.12) computes (for $g \neq 1$) the “curvature densities” $\mathcal{K}_{\mathcal{E}', k}^\pm$ associated to the corresponding Laplacians, $\Delta_{\mathcal{E}', k}^+ = k \partial_{\mathcal{E}'} \partial_{\mathcal{E}'}^*$ on 0-forms, and $\Delta_{\mathcal{E}', k}^- = \partial_{\mathcal{E}'}^* k^2 \partial_{\mathcal{E}'}$ on $(0, 1)$ -forms. This is the analogue of [CoMo14, Theorem 3.2] (which corresponds to $g = 1$) with the distinction that the Morita equivalence trades the conformal metric on the “base” (with Weyl factor $k \in \mathcal{A}_\theta$) for a completely general Hermitian metric on the “bundle”.

Our second result is the extension of [CoMo14, Theorem 4.6] to Heisenberg modules. We use a closed formula for the log-determinant of $\Delta_{\mathcal{E}', k}$ (Theorem 1.15) to prove that the scale invariant version $F_{\mathcal{E}'}$ of the Ray-Singer functional attains its extreme value only if $k = 1$. This means that the extremum is attained for the only Hermitian structure whose associated Hermitian connection has constant curvature (Theorem 1.16). In this way, the Ray-Singer determinant acquires a status similar to that of the Yang-Mills functional of [CoRi87].

Finally, our most noteworthy result (Theorem 1.17) establishes that the gradient at $\log k^2$ of the functional $F_{\mathcal{E}'}$ is equal to the curvature of the conformal metric on T_θ^2 with Weyl factor $k \in \mathcal{A}_\theta$, and thus independent of the Spin^c -structure \mathcal{E}' . Adopting the value of the gradient of the Ray-Singer functional at a metric as the definition of its Gaussian curvature, this proves the invariance of the latter under Morita equivalence. This kind of Morita invariance is a purely noncommutative phenomenon, which in the commutative case passes unnoticed. Nevertheless, the result is somewhat reminiscent of Gauss' *theorem egregium*, if one is willing to liken the metric Spin^c -structures on T_θ^2 to the metrics inherited from embeddings of the ordinary torus in Euclidean space.

The essential technical tool which allows us to obtain the above results is the extension of Connes' pseudodifferential calculus to C^* -dynamical systems on Heisenberg modules. Although quite natural, this extension appears to be of independent interest, in view of other potential applications, such as the computation of the curvature for the Laplacian associated to a Riemannian metric in the sense of J. Rosenberg [Ros13], or more generally that of the index density for an elliptic differential operator on a noncommutative n -torus with coefficients in a Heisenberg module (for the index itself, see [Con80, Theorem 10]).

The other significant technical feature is that we succeed in freeing the entire calculation from any computer assistance. This is accomplished by a systematic

use of known computational shortcuts within the pseudodifferential symbol calculus, supplemented by the manipulation of the modular identities between the 2-parameter family of Laplacians naturally associated with the datum.

We conclude the introduction with a quick outline of the plan of the paper. In Section 1, after a modicum of necessary background, we give the precise formulation of the above mentioned results. The extension of Connes' pseudodifferential calculus to Heisenberg modules, for the general n -dimensional torus, is carried out in Section 2.

Sections 3 and 4 contain the key analytical results of the paper (Theorems 3.1 and 3.2) together with their proofs. As in [CoTr11], the starting point is the recursion formula for the resolvent. We improve the efficiency of the previous calculations in two ways. First, we show that, modulo functions of $\xi = (\xi_1, \xi_2)$ whose average is 0, the relevant coefficient in the asymptotic expansion of the resolvent of the Laplacians can be expressed as a function of $|\eta|^2$, where $\eta = \xi_1 + \bar{\tau}\xi_2$. Secondly, in computing the integral of that coefficient (Section 4) we exploit another kind of symmetry, namely the relations between the conjugates under the action of the Tomita modular operator on the 2-parameter family of Laplacians associated to the twisted spectral triple. This allows to reduce the calculation of the modular curvature functions to the case of the graded Laplacian. Incidentally, it also explains the seemingly magic relation between the two-variable functions H_0 and H_1 in [CoMo14, §3].

The whole resolvent calculation is done in the algebra of pseudodifferential multipliers. Thus, the resulting formulas are universally valid in any effective realization of the pseudodifferential calculus, provided that one is able to relate the operator trace to the dual trace on the multiplier algebra. This is exactly what is done in Section 5, for the Heisenberg representation (Theorems 5.2, 5.3) as well as for the "trivial bundle" case (which was implicitly used in all the previous papers dealing with the subject).

Finally, since the trace formula for a pseudodifferential operator in Connes' calculus is an important technical ingredient for which there is no published proof, we devote Appendix A to a complete derivation of it.

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1. BACKGROUND AND FORMULATION OF MAIN RESULTS

The metric structure of a noncommutative geometric space with coordinate C^* -algebra A is given in spectral terms, by a triplet $(\mathcal{A}, \mathcal{H}, D)$, where \mathcal{A} is a dense, holomorphically closed subalgebra of A , the latter being represented by bounded operators on a Hilbert space \mathcal{H} , and D is an unbounded self-adjoint operator on \mathcal{H} (playing the role of the Dirac operator) such that the commutators $[D, a] = D \circ a - a \circ D$, $a \in \mathcal{A}$, are well-defined and bounded. Local invariants reflecting the curvature of such a space are extracted by means of spectrally defined functionals, from the high frequency behavior of the spectrum of D coupled with the action of the algebra \mathcal{A} on \mathcal{H} . In this section we shall specify these basic notions in the case of the noncommutative torus, and thus provide the necessary background to formulate the main results in precise terms and the appropriate perspective.

1.1. The noncommutative torus T_θ^2 and its standard metric structure. We generally follow the notation in [CoMo14], with some minor deviations. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ be a fixed irrational number. By T_θ^2 we mean the noncommutative space, whose topology is given by the C^* -algebra $A_\theta \equiv C(T_\theta^2)$ generated by two unitaries U_j , $j = 1, 2$, subject to the commutation relation $U_2 U_1 = e^{2\pi i \theta} U_1 U_2$.

1.1.1. Smooth structure. The ordinary torus $T^2 = (\mathbb{R}/\mathbb{Z})^2$ acts on A_θ by the automorphisms $\alpha_r \in \text{Aut}(A_\theta)$, $r \in \mathbb{R}^2$, defined by

$$\alpha_r(U_1^n U_2^m) = e^{2\pi i(r_1 n + r_2 m)} U_1^n U_2^m, \quad r = (r_1, r_2) \in \mathbb{R}^2. \quad (1.1)$$

The smooth structure is given by the subalgebra $C^\infty(T_\theta^2) \equiv \mathcal{A}_\theta \subset A_\theta$ consisting of the smooth elements for the above group of automorphisms *i.e.*, of those $a = \sum_{k,l \in \mathbb{Z}} a_{k,l} U_1^k U_2^l \in A_\theta$ such that the sequence $\{a_{k,l}\} \subset \mathbb{C}$ is rapidly decreasing. The canonical framing of T_θ^2 is given by the infinitesimal generators of the same group of automorphisms *i.e.*, by the derivations $\delta_1, \delta_2 \in \text{Der}(\mathcal{A}_\theta)$ defined by

$$\begin{aligned} \delta_1(U_1) &= 2\pi i U_1, & \delta_1(U_2) &= 0, \\ \delta_2(U_1) &= 0, & \delta_2(U_2) &= 2\pi i U_2; \end{aligned} \quad (1.2)$$

they generate an abelian Lie algebra of derivations $\mathfrak{g}(\mathcal{A}_\theta) := \mathbb{R}\delta_1 + \mathbb{R}\delta_2$.

1.1.2. Standard Spin^c Dirac operator. We fix once and for all a modular parameter $\tau \in \mathbb{C}$ with $\Im \tau > 0$, denote

$$\delta_\tau = \delta_1 + \bar{\tau}\delta_2, \quad \overline{\delta_\tau} = \delta_1 + \tau\delta_2, \quad (1.3)$$

and define a T^2 -invariant complex structure on T_θ^2 via the splitting

$$\mathfrak{g}_\mathbb{C}(\mathcal{A}_\theta) := \mathfrak{g}(\mathcal{A}_\theta) \otimes \mathbb{C} = \mathbb{C}\delta_\tau \oplus \mathbb{C}\overline{\delta_\tau}.$$

We adopt the convention that scalar products are complex antilinear in the first and linear in the second argument. Scalar products will be denoted by (\cdot, \cdot) or $(\cdot, \cdot)_{L^2}$, while the notation $\langle \cdot, \cdot \rangle$ will be reserved for C^* -valued inner products. With φ_0 denoting the unique normalized trace on A_θ , we let $\mathcal{H}_0(\mathcal{A}_\theta) = L^2(\mathcal{A}_\theta, \varphi_0)$ be the completion of \mathcal{A}_θ with respect to the scalar product

$$(a, b)_{L^2} = \varphi_0(a^* b).$$

On the other hand we let $\mathcal{H}^{(1,0)}$ be the unitary bimodule over \mathcal{A}_θ given by the Hilbert space completion of the universal derivation bimodule $\Omega^1(\mathcal{A}_\theta)$ of finite sums $\sum a d(b)$, $a, b \in \mathcal{A}_\theta$ with respect to the inner product

$$(a d(b), a' d(b')) = \varphi_0(a^* a' \delta_\tau(b') \delta_\tau(b)^*), \quad a, a', b, b' \in \mathcal{A}_\theta.$$

We denote by $\Omega^{(1,0)}(\mathcal{A}_\theta) = \{\sum a \partial_\tau(b) \mid a, b \in \mathcal{A}_\theta\}$ the canonical image of $\Omega^1(\mathcal{A}_\theta)$ in $\mathcal{H}^{(1,0)}(\mathcal{A}_\theta)$. Then $\mathcal{A}_\theta \ni a \mapsto \partial_\tau(a) \in \Omega^{(1,0)}(\mathcal{A}_\theta)$ defines an unbounded operator from $\mathcal{H}_0(\mathcal{A}_\theta)$ to $\mathcal{H}^{(1,0)}(\mathcal{A}_\theta)$. By [CoMo14, Lemma 1.5], $\mathcal{H}^{(1,0)}(\mathcal{A}_\theta)$ can be identified with $\mathcal{H}_0(\mathcal{A}_\theta)$, via the map of \mathcal{A}_θ -modules $\kappa : \mathcal{H}^{(1,0)}(\mathcal{A}_\theta) \rightarrow \mathcal{H}_0(\mathcal{A}_\theta)$ extending the assignment

$$\Omega^{(1,0)}(\mathcal{A}_\theta) \ni \sum a \partial_\tau(b) \mapsto \kappa\left(\sum a \partial_\tau(b)\right) := \sum a \delta_\tau(b) \in \mathcal{H}_0(\mathcal{A}_\theta). \quad (1.4)$$

Under this identification, ∂_τ becomes the derivation δ_τ , viewed as unbounded operator from $\mathcal{H}_0(\mathcal{A}_\theta)$ to $\mathcal{H}^{(1,0)}(\mathcal{A}_\theta)$.

The Spin^c Dirac operator associated to the fixed complex structure and to the flat metric on T_θ^2 is the $(\partial_\tau + \partial_\tau^*)$ -operator

$$D = \begin{pmatrix} 0 & \partial_\tau^* \\ \partial_\tau & 0 \end{pmatrix} \quad \text{acting on} \quad \tilde{\mathcal{H}} = \mathcal{H}_0(\mathcal{A}_\theta) \oplus \mathcal{H}^{(1,0)}(\mathcal{A}_\theta), \quad (1.5)$$

which is isospectral to the usual Spin^c Dirac operator on the complex torus \mathbb{C}/Γ , $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$. Both the left and the right action of \mathcal{A}_θ on $\tilde{\mathcal{H}}$ are unitary and give rise to spectral triples, $(\mathcal{A}_\theta, \tilde{\mathcal{H}}, D)$ and $(\mathcal{A}_\theta^{\text{op}}, \tilde{\mathcal{H}}, D)$. The transposed of the latter in the sense of [CoMo14, Def. 1.4] is isomorphic to the former, but with the opposite grading.

Remark 1.1. Note that the space of 1-forms corresponding to the first spectral triple, $\Omega_D(\mathcal{A}_\theta) = \{\sum a[D, b]; a, b \in \mathcal{A}_\theta\}$, is isomorphic as \mathcal{A}_θ -bimodule to the direct sum $\Omega^{(1,0)}(\mathcal{A}_\theta) \oplus \Omega^{(0,1)}(\mathcal{A}_\theta)$, where $\Omega^{(0,1)}(\mathcal{A}_\theta) := \{\sum a \partial_\tau^*(b); a, b \in \mathcal{A}_\theta\}$.

1.2. Heisenberg modules and their holomorphic structures. As mentioned in the introduction, we view the basic Heisenberg modules over the noncommutative space T_θ^2 as analogues of the fundamental complex spinor bundles over T^2 . We recall below their definition as well as their main features.

1.2.1. Basic Heisenberg modules. We generally adopt the notation in [PoSco3], except for a few changes which makes it compatible with Rieffel's general construction [Rie88, Sec. 2,3]. For the convenience of the reader we have collected a synopsis of the latter in the Appendix A. To make the connection between the Appendix and the following material see in particular the Example A.6.

Fix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{Z})$, and let $g\theta = \frac{a\theta+b}{c\theta+d}$. In the sequel we use the abbreviation θ' for $g\theta$. Unless otherwise specified, it will be assumed that $c \neq 0$. For an explanation of the following formulas see Section A.6.

Let $\mathcal{E}(g, \theta)$ be the Schwartz space $\mathcal{S}(\mathbb{R})^{|\mathbb{Z}|} \equiv \mathcal{S}(\mathbb{R} \times \mathbb{Z}_c)$, with $\mathbb{Z}_c := \mathbb{Z}/c\mathbb{Z}$. The (smooth) algebra \mathcal{A}_θ acts on the right by

$$(fU_1)(t, \alpha) := e^{2\pi i(t - \frac{\alpha d}{c})} f(t, \alpha), \quad (fU_2)(t, \alpha) := f(t - \frac{c\theta + d}{c}, \alpha - 1); \quad (1.6)$$

the algebra $\mathcal{A}_{\theta'}$, with generators denoted V_1, V_2 , acts on the left by

$$(V_1 f)(t, \alpha) := e^{2\pi i \left(\frac{t}{c\theta+d} - \frac{\alpha}{c} \right)} f(t, \alpha), \quad (V_2 f)(t, \alpha) := f\left(t - \frac{1}{c}, \alpha - a\right). \quad (1.7)$$

By analogy with vector bundles over an elliptic curve, one defines the rank, degree and slope of $\mathcal{E}(g, \theta)$ as the numbers

$$\begin{aligned} \text{rk } \mathcal{E}(g, \theta) &= c\theta + d, \quad \deg \mathcal{E}(g, \theta) = c, \text{ resp.} \\ \mu(\mathcal{E}(g, \theta)) &:= \frac{\deg \mathcal{E}(g, \theta)}{\text{rk } \mathcal{E}(g, \theta)} = \frac{1}{\theta + \frac{d}{c}}. \end{aligned} \quad (1.8)$$

For $f_1, f_2 \in \mathcal{E}(g, \theta)$ let

$$(f_1, f_2)_{L^2} := \int_{\mathbb{R} \times \mathbb{Z}_c} \overline{f_1(t, \alpha)} f_2(t, \alpha) dt d\alpha \quad (1.9)$$

denote the L^2 -scalar product, where the integration is with respect to the Lebesgue measure on \mathbb{R} and the counting measure on \mathbb{Z}_c . This determines an \mathcal{A}_θ -valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}_\theta} : \mathcal{E}(g, \theta) \times \mathcal{E}(g, \theta) \rightarrow \mathcal{A}_\theta$ (antilinear in the first argument), and also an $\mathcal{A}_{\theta'}$ -valued inner product ${}_{\mathcal{A}_{\theta'}} \langle \cdot, \cdot \rangle : \mathcal{E}(g, \theta) \times \mathcal{E}(g, \theta) \rightarrow \mathcal{A}_{\theta'}$ (antilinear in the second argument), both uniquely characterized by the identity

$$|c\theta + d| \varphi'_0({}_{\mathcal{A}_{\theta'}} \langle f_2, f_1 \rangle) = (f_1, f_2)_{L^2} = \varphi_0(\langle f_1, f_2 \rangle_{\mathcal{A}_\theta}), \quad (1.10)$$

cf. [PoSco3, Sec. 1].

The completion $E(g, \theta)$ of $\mathcal{E}(g, \theta)$ with respect to $\|\langle \cdot, \cdot \rangle_{\mathcal{A}_\theta}\|^{1/2}$ is a full right C^* -module over \mathcal{A}_θ , whose endomorphism algebra is $\text{End}_{\mathcal{A}_\theta}(E(g, \theta)) = \mathcal{A}_{\theta'}$; at the same time, it is a full left C^* -module over $\mathcal{A}_{\theta'}$, and $\text{End}_{\mathcal{A}_{\theta'}}(E(g, \theta)) = \mathcal{A}_\theta$.

On the other hand, the completion $\mathcal{H}_0(g, \theta)$ of $\mathcal{E}(g, \theta)$ with respect to the scalar product Eq. (1.9) is the Hilbert space $L^2(\mathbb{R} \times \mathbb{Z}_c) = L^2(\mathbb{R})^{|\mathbb{Z}_c|}$. Note that the Hilbert space $\mathcal{H}_0(g, \theta)$ also coincides with the interior tensor product $E(g, \theta) \otimes_{\mathcal{A}_\theta} L^2(\mathcal{A}_\theta, \varphi_0)$.

The bimodules $\mathcal{E}(g, \theta)$, $\theta \in (0, 1)$, $g \in GL(2, \mathbb{Z})$, which are called *basic*, correspond to pairs of relatively prime integers (d, c) . A similar construction obviously applies to any pair $(d, c) \in \mathbb{Z}^2$, but the resulting bimodule is isomorphic to a direct sum of m copies of the module associated to the pair of relatively prime integers $(\frac{d}{m}, \frac{c}{m})$, where m is the greatest common divisor of the pair (d, c) .

The standard connection $\nabla^\mathcal{E}$ on $\mathcal{E} = \mathcal{E}(g, \theta)$ is given by the directional covariant derivatives $\nabla_j : \mathcal{E}(g, \theta) \rightarrow \mathcal{E}(g, \theta)$, $j = 1, 2$, defined by

$$(\nabla_1 f)(t, \alpha) := \frac{\partial}{\partial t} f(t, \alpha), \quad (\nabla_2 f)(t, \alpha) := 2\pi i \cdot \mu(\mathcal{E}(g, \theta)) \cdot t \cdot f(t, \alpha). \quad (1.11)$$

It is actually a bimodule connection, in the sense that it is compatible with the standard derivations (δ_1, δ_2) on \mathcal{A}_θ , as well as with the (normalized) derivations $\delta'_j := \frac{1}{\text{rk } \mathcal{E}(g, \theta)} \delta_j$, $j = 1, 2$, on $\mathcal{A}_{\theta'}$. Specifically, for $f \in \mathcal{E}(g, \theta)$, $a \in \mathcal{A}_{\theta'}$, $b \in \mathcal{A}_\theta$, one has

$$\nabla_j(a \cdot f \cdot b) = a \cdot (\nabla_j f) \cdot b + \delta'_j(a) \cdot f \cdot b + a \cdot f \cdot \delta_j(b), \quad j = 1, 2. \quad (1.12)$$

Furthermore, the connection is *Hermitian*, meaning that

$$\begin{aligned}\delta_j(\langle f_1, f_2 \rangle_{\mathcal{A}_\theta}) &= \langle \nabla_j f_1, f_2 \rangle_{\mathcal{A}_\theta} + \langle f_1, \nabla_j f_2 \rangle_{\mathcal{A}_\theta}, \\ \delta'_j(\langle f_1, f_2 \rangle_{\mathcal{A}_\theta}) &= \langle \nabla_j f_1, f_2 \rangle_{\mathcal{A}_\theta} + \langle f_1, \nabla_j f_2 \rangle_{\mathcal{A}_\theta}.\end{aligned}\tag{1.13}$$

The bimodules of the form $\mathcal{E}(g, \theta)$ are called *basic Heisenberg modules*. The designation “Heisenberg modules” comes from the fact that the standard connection satisfies the Heisenberg commutation relation

$$[\nabla_1, \nabla_2] = 2\pi i \cdot \mu(\mathcal{E}(g, \theta)) \cdot \text{Id},\tag{1.14}$$

which expresses the fact that it has *constant curvature*.

This relation is the infinitesimal form of a unitary representation of the (polarized) Heisenberg group $H = \mathbb{R}^2 \times \mathbb{R}$ with multiplication law

$$(x, \zeta) \cdot (y, \eta) := (x + y, \zeta + \eta + \sigma(x, y)),\tag{1.15}$$

where $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the standard symplectic form

$$\sigma(x, y) := x_1 y_2 - x_2 y_1, \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.\tag{1.16}$$

Indeed, one easily checks that the formula

$$(\pi_0(x, \zeta)f)(t) := e^{\pi i \mu(\mathcal{E}(g, \theta))(\zeta + x_1 x_2 + 2x_2 t)} f(t + x_1), \quad f \in L^2(\mathbb{R}),\tag{1.17}$$

defines a unitary representation of H on $L^2(\mathbb{R})$. This is in fact the unique (up to unitary equivalence) representation of H with central character

$$\text{Center}(H) \ni (0, \zeta) \mapsto e^{\pi i \mu(\mathcal{E}(g, \theta))\zeta} \in \mathbb{T}.$$

By a slight abuse, we shall continue to use the same notation for its multiple $\pi_0 \otimes \text{Id}$ acting on the Hilbert space $L^2(\mathbb{R} \times \mathbb{Z}_c) = L^2(\mathbb{R}) \otimes \mathbb{C}^{|\mathbb{Z}_c|}$. The space of C^∞ -vectors of the latter is precisely $\mathcal{E}(g, \theta)$.

For later use, let us point out that there is a 2-parameter family of unitarily equivalent representations to π_0 . Indeed, associating to each $w \in \mathbb{R}^2$ the unitary character $\chi_w : H \rightarrow \mathbb{T}$,

$$\chi_w(x, \zeta) := e^{i\langle w, x \rangle},$$

one forms the interior tensor product representation

$$\pi_w := \chi_w \otimes \pi_0,\tag{1.18}$$

which has the same central character as π_0 . It will be convenient to treat this family as projective representations of \mathbb{R}^2 , $\pi_w(x) := \pi_w(x, 0)$, satisfying the cocycle identity

$$\pi_w(x)\pi_w(y) = e^{\pi i \mu(\mathcal{E}(g, \theta))\sigma(x, y)} \pi_w(x + y).\tag{1.19}$$

At the infinitesimal level one has

$$\begin{aligned}\frac{\partial}{\partial x_j} \Big|_{x=0} \pi_w(x) &= \nabla_j + i w_j, & \text{analogous to} \\ \frac{\partial}{\partial x_j} \Big|_{x=0} \alpha_x &= \delta_j, & \frac{\partial}{\partial x_j} \Big|_{x=0} \alpha'_x = \delta'_j, \quad j = 1, 2.\end{aligned}\tag{1.20}$$

In particular, Eq. (1.11) and Eq. (1.12) follow from Eq. (1.20). We also note that π_w implements the action of T^2 on the $\mathcal{A}_{\theta'}$ and \mathcal{A}_θ , i.e., satisfies

$$\begin{aligned}\pi_w(x) V_j \pi_w(x)^* &= e^{\frac{2\pi i}{\tau k \mathcal{E}(g, \theta)} x_j} V_j =: \alpha'_x(V_j), \\ \pi_w(x) U_j \pi_w(x)^* &= e^{2\pi i x_j} U_j =: \alpha_x(U_j).\end{aligned}\tag{1.21}$$

1.2.2. Spin^c Dirac operators with coefficients in Heisenberg modules. Given a Heisenberg module as above $\mathcal{E} = \mathcal{E}(g, \theta)$ with $c > 0$, and equipped with its standard connection $\nabla^\mathcal{E}$, one can apply the general recipe (cf. [CON94, VI.1]) to form the twisted version of the standard Spin^c Dirac operator $D = \partial_\tau + \partial_\tau^*$ with coefficients in \mathcal{E} :

$$\begin{aligned}D_\mathcal{E} &= \begin{pmatrix} 0 & D_\mathcal{E}^- \\ D_\mathcal{E}^+ & 0 \end{pmatrix}, \quad \text{where } D_\mathcal{E}^- = (D_\mathcal{E}^+)^* \\ \text{and } D_\mathcal{E}^+(f \otimes a) &= \partial_\tau(f) a + f \otimes \partial_\tau(a), \quad f \in \mathcal{E}, a \in \mathcal{A}_\theta.\end{aligned}\tag{1.22}$$

Here the standard connection is regarded as a map $\nabla^\mathcal{E} : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}_\theta} \Omega_D^1(\mathcal{A}_\theta)$. In view of the decomposition of 1-forms from Remark 1.1, combined with the canonical identification (see [CON94, VI.3, Lemma 12]) of $\Omega_D^1(\mathcal{A}_\theta)$ with off-diagonal matrices in $M_2(\mathcal{A}_\theta)$, the connection $\nabla^\mathcal{E}$ on $\mathcal{E} = \mathcal{E}(g, \theta)$ splits into a *holomorphic* and an *anti-holomorphic* component

$$\nabla^\mathcal{E} = \partial_\mathcal{E} \oplus \partial_\mathcal{E}^*, \quad \text{where } \partial_\mathcal{E} := \nabla_1 + \bar{\tau} \nabla_2.\tag{1.23}$$

The operator $\partial_\mathcal{E} : \Omega^0(\mathcal{E}) \rightarrow \Omega^{(1,0)}(\mathcal{E}) = \mathcal{E} \otimes_{\mathcal{A}_\theta} \Omega^{(1,0)}(\mathcal{A}_\theta)$ defines a *holomorphic structure*, i.e., satisfies the property

$$\partial_\mathcal{E}(fa) = \partial_\mathcal{E}(f) a + f \partial_\tau(a), \quad f \in \mathcal{E}, a \in \mathcal{A}_\theta.\tag{1.24}$$

Together with the identity in the second line of Eq. (1.22), which can be equivalently written as

$$D_\mathcal{E}^+(f a \otimes 1) = \partial_\mathcal{E}(f) a + f \partial_\tau(a) \otimes 1, \quad f \in \mathcal{E}, a \in \mathcal{A}_\theta,$$

this shows that $D_\mathcal{E}^+$ is just the extension of $\partial_\mathcal{E}$ to an unbounded operator from $\mathcal{H}_0(g, \theta)$ to $\mathcal{H}^{(1,0)}(g, \theta) := E(g, \theta) \otimes_{\mathcal{A}_\theta} \mathcal{H}^{(1,0)}(\mathcal{A}_\theta)$. The latter can be identified to $\mathcal{H}_0(g, \theta)$ via the bimodule isomorphism

$$\kappa_{g, \theta} = \text{Id} \otimes_{\mathcal{A}_\theta} \kappa : E(g, \theta) \otimes_{\mathcal{A}_\theta} \mathcal{H}^{(1,0)}(\mathcal{A}_\theta) \rightarrow \mathcal{H}_0(g, \theta) = E(g, \theta) \otimes_{\mathcal{A}_\theta} \mathcal{H}_0(\mathcal{A}_\theta).\tag{1.25}$$

It follows that

$$D_\mathcal{E} = \begin{pmatrix} 0 & \partial_\mathcal{E}^* \\ \partial_\mathcal{E} & 0 \end{pmatrix} \quad \text{acting on } \tilde{\mathcal{H}}(g, \theta) = \mathcal{H}_0(g, \theta) \oplus \mathcal{H}^{(1,0)}(g, \theta),\tag{1.26}$$

giving a spectral triple $(A_\theta^{\text{op}}, \tilde{\mathcal{H}}(g, \theta), D_\mathcal{E})$ for the right action of \mathcal{A}_θ .

As a matter of fact, the holomorphic component $\partial_\mathcal{E}$ satisfies the analogous identity to Eq. (1.12),

$$\partial_\mathcal{E}(afb) = a(\partial_\mathcal{E}f)b + \delta'_\tau(a)fb + a f \delta_\tau(b), \quad f \in \mathcal{E}, a \in \mathcal{A}'_\theta, b \in \mathcal{A}_\theta,\tag{1.27}$$

where $\delta_\tau = \delta_1 + \bar{\tau} \delta_2$ and $\delta'_\tau = \delta'_1 + \bar{\tau} \delta'_2$.

Such a map is called in [PoSco3] a *standard holomorphic structure*. There is a one-parameter family of such structures, given by

$$\partial_{\mathcal{E},z} := \partial_{\mathcal{E}} + z \cdot \text{Id}, \quad z \in \mathbb{C}. \quad (1.28)$$

It turns out that these are the only ones. Actually, an even stronger statement holds true.

Proposition 1.2. *Assume $g \in \text{GL}(2, \mathbb{Z})$ with $c \neq 0$ and let $\tilde{\partial}$ be a holomorphic structure on $\mathcal{E}(g, \theta)$ which is compatible with some $\tilde{\delta} \in \mathfrak{g}(\mathcal{A}_{\theta'})$, i.e., satisfies*

$$\tilde{\partial}(af) = a\tilde{\partial}(f) + \tilde{\delta}(a)f, \quad f \in \mathcal{E}(g, \theta), a \in \mathcal{A}_{\theta'}.$$

Then $\tilde{\delta} = \delta'_{\tau} = \delta'_1 + \bar{\tau}\delta'_2$ and $\nabla = \partial_{\mathcal{E},z}$ for some $z \in \mathbb{C}$.

Proof. In view of Eq. (1.27), the difference $\tilde{\partial} - \partial_{\mathcal{E}}$ is \mathcal{A}_{θ} -linear, hence it is given by the left action of some $\omega \in \mathcal{A}_{\theta'}$. Let us show that $\tilde{\partial} - \partial_{\mathcal{E}}$ is also $\mathcal{A}_{\theta'}$ -linear. As $\tilde{\partial} = \partial_{\mathcal{E}} + \omega \cdot$, for any $a \in \mathcal{A}_{\theta'}$, $f \in \mathcal{E}(g, \theta)$ we have on the one hand,

$$\tilde{\partial}(af) = \tilde{\delta}(a)f + a\partial_{\mathcal{E}}f + a\omega f, \quad (1.29)$$

and on the other hand

$$\tilde{\partial}(af) = \partial_{\mathcal{E}}(af) + \omega af = a\partial_{\mathcal{E}}f + \delta'_{\tau}(a)f + [\omega, a]f + a\omega f. \quad (1.30)$$

This implies $\tilde{\delta} = \delta'_{\tau} + [\omega, \cdot]$. Hence $\tilde{\delta} - \delta'_{\tau}$ is an inner derivation of $\mathcal{A}_{\theta'}$. Since it is also a linear combination of δ'_j , it follows that $\tilde{\delta} = \delta'_{\tau}$. But then ω is in the center of $\mathcal{A}_{\theta'}$, and so $\omega \in \mathbb{C} \cdot \text{Id}$. \square

Remark 1.3. The property Eq. (1.27) ensures that the operator $D_{\mathcal{E}}$ also gives rise to a spectral triple $(\mathcal{A}_{\theta'}, \tilde{\mathcal{H}}(g, \theta), D_{\mathcal{E}})$ for the left action of $\mathcal{A}_{\theta'}$.

Remark 1.4. Using the family of holomorphic structures Eq. (1.28) one could define a family of operators $\{D_{\mathcal{E},z}; z \in \mathbb{C}\}$. The corresponding spectral triples would not be essentially different however, because $D_{\mathcal{E},z}$ is merely an internal perturbation (in the sense of [Con96]) of the operator $D_{\mathcal{E}}$.

Proposition 1.5. *Assume $\Im(\tau) > 0$ and $\deg(\mathcal{E}) \neq 0$.*

- (1) *If $\mu(\mathcal{E}) > 0$ then $\dim \text{Ker } \partial_{\mathcal{E}} = |\deg(\mathcal{E})|$ and $\text{Ker } \partial_{\mathcal{E}}^* = 0$; if $\mu(\mathcal{E}) < 0$ then $\dim \text{Ker } \partial_{\mathcal{E}}^* = |\deg(\mathcal{E})|$ and $\text{Ker } \partial_{\mathcal{E}} = 0$.*
- (2) *The zeta function $\zeta_{\Delta_{\mathcal{E}}}(s) = \text{Tr}(\Delta_{\mathcal{E}}^{-s})$, $\Re(s) > 1$, has a meromorphic continuation to \mathbb{C} ; it is regular at $s = 0$ and one has*

$$\zeta_{\Delta_{\mathcal{E}}}(0) = -\frac{1}{2}|\deg(\mathcal{E})|, \quad (1.31)$$

$$\zeta'_{\Delta_{\mathcal{E}}}(0) = \frac{1}{2}|\deg(\mathcal{E})| \cdot \log(2|\mu(\mathcal{E})|\Im(\tau)), \quad (1.32)$$

$$\text{Res}_{s=1} \zeta_{\Delta_{\mathcal{E}}}(s) = \frac{|\text{rk}(\mathcal{E})|}{4\pi\Im(\tau)}. \quad (1.33)$$

Proof. Claim (1), also proved in [PoSco3, Proof of Prop. 2.5], is easy to justify. By its very definition, $\partial_{\mathcal{E}}$ is a direct sum of $|\deg(\mathcal{E})|$ copies of the operator $D = \frac{d}{dt} + 2\pi i\mu(\mathcal{E})\bar{\tau}t$. The latter has 1-dimensional kernel in $\mathcal{S}(\mathbb{R})$ and no cokernel when $\mu(\mathcal{E}) > 0$, since then $\Re(2\pi i\mu(\mathcal{E})\bar{\tau}) = 2\pi\mu(\mathcal{E})\Im(\tau) > 0$.

When $\mu(\mathcal{E}) < 0$ the same argument applies to the operator $\partial_{\mathcal{E}}^*$.

Claim (2) also follows from a routine calculation involving the harmonic oscillator. Indeed, the Laplacian $\Delta_{\mathcal{E}} = \partial_{\mathcal{E}}^* \partial_{\mathcal{E}}$ is a direct sum of $|\deg(\mathcal{E})|$ copies of the harmonic oscillator

$$H := D^* D = -\frac{d^2}{dt^2} + 4\pi^2 \mu(\mathcal{E})^2 |\tau|^2 t^2 - 4\pi i \mu(\mathcal{E}) \Re(\tau) t \frac{d}{dt} - 2\pi i \mu(\mathcal{E}) \bar{\tau} \text{Id}. \quad (1.34)$$

One easily checks that $[D, D^*] = 4\pi \mu(\mathcal{E}) \Im(\tau) \text{Id}$, and so $DD^* = H + 4\pi \mu(\mathcal{E}) \Im(\tau) \text{Id}$, which implies

$$HD^* = D^*(DD^*) = D^*(H + 4\pi \mu(\mathcal{E}) \Im(\tau) \text{Id}). \quad (1.35)$$

On the other hand, $DH = (DD^*)D = (H + 4\pi \mu(\mathcal{E}) \Im(\tau) \text{Id})D$, whence

$$HD = D(H - 4\pi \mu(\mathcal{E}) \Im(\tau) \text{Id}). \quad (1.36)$$

Thus, by Eq. (1.35) D^* shifts forward each eigenspace V_{λ} of H to $V_{\lambda+4\pi \mu(\mathcal{E}) \Im(\tau)}$, while by Eq. (1.36) D shifts backward V_{λ} onto $V_{\lambda-4\pi \mu(\mathcal{E}) \Im(\tau)}$. Since $\dim(\text{Ker } D) = 1$ for $\mu(\mathcal{E}) > 0$ resp. $\dim(\text{Ker } D^*) = 1$ for $\mu(\mathcal{E}) < 0$, one concludes that the spectrum of H is

$$\text{spec } H = 4\pi |\mu(\mathcal{E})| \Im(\tau) \begin{cases} \mathbb{Z}_+, & \mu(\mathcal{E}) > 0, \\ \mathbb{Z}_+ \setminus \{0\}, & \mu(\mathcal{E}) < 0, \end{cases}$$

and each eigenvalue has multiplicity 1. In any case, the ζ -function is

$$\zeta_{\Delta_{\mathcal{E}}}(s) = |\deg(\mathcal{E})| \cdot (4\pi |\mu(\mathcal{E})| \Im(\tau))^{-s} \cdot \zeta_{\mathbb{R}}(s)$$

with the Riemann zeta function $\zeta_{\mathbb{R}}(s) = \sum_{n=1}^{\infty} n^{-s}$.

The equations Eq. (1.31), (1.32) and (1.33) immediately follow from the corresponding values of the Riemann zeta function, $\text{Res}_{s=1} \zeta_{\mathbb{R}}(s) = 1$, $\zeta_{\mathbb{R}}(0) = -\frac{1}{2}$ and $\zeta'_{\mathbb{R}}(0) = -\frac{1}{2} \log(2\pi)$. \square

1.3. Conformal twisting and curvature.

1.3.1. Conformal change of background metric. We begin by constructing, in the manner of [CoMo14, §1.5], a bi-spectral triple associated to a ‘noncommutative Weyl factor’ $k = e^{h/2} \in \mathcal{A}_{\theta}$ with $h = h^* \in \mathcal{A}_{\theta}$.

Let $\varphi = \varphi_h$ be defined by

$$\varphi(a) = \varphi_0(ae^{-h}), \quad a \in \mathcal{A}_{\theta}$$

and let $\mathcal{H}_{\varphi} = L^2(\mathcal{A}_{\theta}, \varphi)$ be the Hilbert space completion of \mathcal{A}_{θ} with respect to the inner product

$$(a, b) = \varphi(a^* b), \quad a, b \in \mathcal{A}_{\theta}.$$

Denote by $\mathcal{H}_{\mathcal{E}, \varphi} = \mathcal{H}_{\varphi}(g, \theta)$ the interior tensor product $E(g, \theta) \otimes_{\mathcal{A}_{\theta}} L^2(\mathcal{A}_{\theta}, \varphi)$ and by $\mathcal{H}_{\mathcal{E}}^{(1,0)} = \mathcal{H}_0(g, \theta)$ the interior tensor product $E(g, \theta) \otimes_{\mathcal{A}_{\theta}} \mathcal{H}_0$, where $\mathcal{H}_0 = L^2(\mathcal{A}_{\theta}, \varphi_0)$. Both are $(\mathcal{A}_{\theta}', \mathcal{A}_{\theta})$ -bimodules but the natural (right) action of \mathcal{A}_{θ} on $\mathcal{H}_{\mathcal{E}, \varphi}$ is no longer unitary. To rectify this, one replaces it, as in [CoMo14, §1.5], by the unitary action

$$a^{\text{op}}(f) = f \cdot k^{-1} a k, \quad a \in \mathcal{A}_{\theta}, f \in \mathcal{E}(g, \theta).$$

In the present setting, Lemma 1.7 in [CoMo14] has the following counterpart:

Lemma 1.6. *The right action of $k = e^{h/2}$ on $\mathcal{E}(g, \theta)$,*

$$W(f) = f \cdot k, \quad f \in \mathcal{E}(g, \theta),$$

extends to an isometry $W : \mathcal{H}_0(g, \theta) \rightarrow \mathcal{H}_{\mathcal{E}, \varphi}$, which establishes an isometric $(\mathcal{A}_{\theta'}, \mathcal{A}_{\theta})$ -bimodule isomorphism between $\mathcal{H}_0(g, \theta)$ and $\mathcal{H}_{\mathcal{E}, \varphi}$.

Proof. Indeed, if $f, g \in \mathcal{E}(g, \theta)$ then

$$(W(f), W(g))_{\mathcal{H}_{\mathcal{E}, \varphi}} = \varphi(\langle g \cdot k, f \cdot k \rangle_{\mathcal{A}_{\theta}}) = \varphi_0(\langle f, g \rangle_{\mathcal{A}_{\theta}}) = (f, g)_{\mathcal{H}_0(g, \theta)}.$$

By its very definition W is $\mathcal{A}_{\theta'}$ -linear. On the other hand, for any $a \in \mathcal{A}_{\theta}$

$$a^{\text{op}}(W(f)) = f \cdot k \cdot k^{-1} a k = W(f \cdot a). \quad \square$$

Let now $\partial_{\mathcal{E}, \varphi}$ be the closure of $\partial_{\mathcal{E}}$ viewed as unbounded operator from $\mathcal{E}(g, \theta) \subset \mathcal{H}_{\varphi}(g, \theta)$ to $\mathcal{H}^{(1,0)}(g, \theta)$. The analogue of [CoMo14, Cor. 1.9 (ii)] reads as follows.

Proposition 1.7. *The operator*

$$D_{\mathcal{E}, \varphi} = \begin{pmatrix} 0 & \partial_{\mathcal{E}, \varphi}^* \\ \partial_{\mathcal{E}, \varphi} & 0 \end{pmatrix} \quad \text{acting on} \quad \tilde{\mathcal{H}}_{\mathcal{E}, \varphi} = \mathcal{H}_{\mathcal{E}, \varphi} \oplus \mathcal{H}^{(1,0)}(g, \theta) \quad (1.37)$$

gives rise to a twisted graded spectral triple $(A_{\theta}^{\text{op}}, \tilde{\mathcal{H}}_{\mathcal{E}, \varphi}, D_{\mathcal{E}, \varphi})$ with respect to the right action of A_{θ} . On the other hand the left action of $A_{\theta'}$ yields an ordinary graded spectral triple $(A_{\theta'}, \tilde{\mathcal{H}}_{\mathcal{E}, \varphi}, D_{\mathcal{E}, \varphi})$.

Proof. The twisting by the automorphism $\sigma \in \text{Aut}(A_{\theta})$, $\sigma(a) = k^{-1} a k$, becomes self-evident when the operator $D_{\mathcal{E}, \varphi}$ is transferred via the isometry W . Indeed, denoting $\widetilde{W} = W \oplus \kappa_{g, \theta}^{-1}$, one has

$$\widetilde{W}^* D_{\mathcal{E}, \varphi} \widetilde{W} = \begin{pmatrix} 0 & R_k \partial_{\mathcal{E}}^* \\ \partial_{\mathcal{E}} R_k & 0 \end{pmatrix} \quad \text{acting on} \quad \tilde{\mathcal{H}}(g, \theta) = \mathcal{H}_0(g, \theta) \oplus \mathcal{H}_0(g, \theta), \quad (1.38)$$

□

Lemma 1.8. *For $g \in \text{GL}(2, \mathbb{Z})$ with $c \neq 0$ the map defined by*

$$J_{g, \theta}(f)(x, \alpha) = \overline{f((c\theta + d)x, -d^{-1}\alpha)} \quad (1.39)$$

is an antiisomorphism of C^ -bimodules $\mathcal{E}(g, \theta) \rightarrow \mathcal{E}(g^{-1}, g\theta)$. More precisely, $J = J_{g, \theta}$ satisfies the following identities for $f, f_1, f_2 \in \mathcal{E}(g, \theta)$, $a \in \mathcal{A}_{\theta'}$, $b \in \mathcal{A}_{\theta}$:*

$$J(afb) = b^* J(f) a^*, \quad (1.40)$$

$${}_{\mathcal{A}_{\theta}} \langle J(f_1), J(f_2) \rangle = \langle f, g \rangle_{\mathcal{A}_{\theta}}, \quad (1.41)$$

$$\langle J(f_1), J(f_2) \rangle_{\mathcal{A}_{\theta'}} = {}_{\mathcal{A}_{\theta'}} \langle f_1, f_2 \rangle, \quad (1.42)$$

$$|c\theta + d| (J(f_1), J(f_2))_{L^2} = (f_2, f_1)_{L^2}. \quad (1.43)$$

In addition, $J_{g, \theta}^{-1} = J_{g^{-1}, g\theta}$.

Proof. This is checked by direct calculation. Note that Eq. (1.42), (1.43) follow from Eq. (1.40) and (1.41). □

Proposition 1.9. *The transposed of the twisted spectral triple $(A_\theta^{\text{op}}, \tilde{\mathcal{H}}(g, \theta), D_{\mathcal{E}, \varphi})$ is isomorphic to the twisted spectral triple $(A_\theta, \mathcal{H}_0(g^{-1}, \theta') \oplus \mathcal{H}_0(g^{-1}, \theta'), \overline{D}_{\mathcal{E}', k})$ where*

$$\overline{D}_{\mathcal{E}', k} = -\text{rk}(\mathcal{E}') \begin{pmatrix} 0 & k\partial_{\mathcal{E}'} \\ \partial_{\mathcal{E}'}^* & 0 \end{pmatrix}, \quad \text{with } \partial_{\mathcal{E}'} = \nabla'_1 + \overline{\tau}\nabla'_2. \quad (1.44)$$

In turn, the spectral triple $(A_{\theta'}, \tilde{\mathcal{H}}_{\mathcal{E}, \varphi}, D_{\mathcal{E}, \varphi})$ is isomorphic to the spectral triple $(A_{\theta'}^{\text{op}}, \mathcal{H}_0(g^{-1}, \theta') \oplus \mathcal{H}_0(g^{-1}, \theta'), \overline{D}_{\mathcal{E}', k})$.

Proof. The conjugate of the operator $\partial_{\mathcal{E}}$ by the above antiunitary isometry is related to $\partial_{\mathcal{E}'}$ by the identity (see [Pol04, p. 175]):

$$J_{g, \theta} \circ \partial_{\mathcal{E}}^* \circ J_{g, \theta}^{-1} = -\frac{1}{\text{rk} \mathcal{E}(g, \theta)} \partial_{\mathcal{E}'}. \quad (1.45)$$

Setting $\tilde{J} = J_{g, \theta} \oplus (-J_{g, \theta})$, it follows that

$$\tilde{J} \tilde{W}^* D_{\mathcal{E}, \varphi} \tilde{W} \tilde{J}^{-1} = -\frac{1}{\text{rk} \mathcal{E}(g, \theta)} \begin{pmatrix} 0 & k\partial_{\mathcal{E}'} \\ \partial_{\mathcal{E}'}^* & 0 \end{pmatrix}. \quad (1.46)$$

It remains to notice that $\frac{1}{\text{rk} \mathcal{E}(g, \theta)} = \text{rk}(\mathcal{E}')$. \square

The Heisenberg module underlying the above bi-spectral triple is the $(\mathcal{A}_\theta, \mathcal{A}_{\theta'})$ -bimodule $\mathcal{E}' = \mathcal{E}(g^{-1}, \theta')$, with \mathcal{A}_θ identified to $\text{End}_{\mathcal{A}_{\theta'}}(\mathcal{E}')$. Moreover, as can be seen from Eq. (1.10), in this picture turning on the Weyl factor $k = e^{\frac{h}{2}} \in \mathcal{A}_\theta$ amounts to passing to the Hermitian structure

$$(f'_1, f'_2)_k := \langle k f'_1, f'_2 \rangle_{\mathcal{A}_\theta}, \quad f'_1, f'_2 \in \mathcal{E}'. \quad (1.47)$$

It will be shown below that such a change uniquely determines a holomorphic connection on \mathcal{E}' which is Hermitian with respect to the new metric. However, for notational convenience, we momentarily switch the roles of the two Morita equivalent algebras and phrase the uniqueness result in terms of the original Heisenberg $(\mathcal{A}_{\theta'}, \mathcal{A}_\theta)$ -bimodule $\mathcal{E} = \mathcal{E}(g, \theta)$.

Proposition 1.10. *Let $K \in \text{End}_{\mathcal{A}_\theta}(\mathcal{E}) = \mathcal{A}_{\theta'}$ be positive definite and consider the inner product*

$$H(f, g) := \langle Kf, g \rangle_{\mathcal{A}_\theta}. \quad (1.48)$$

Then there is a unique connection ∇^K on \mathcal{E} such that

$$\partial^K = \nabla_1^K + \overline{\tau}\nabla_2^K \quad \text{is a holomorphic structure,} \quad (1.49)$$

and which is Hermitian with respect to H , i.e., satisfies

$$\delta_j H(f_1, f_2) = H(\nabla_j^K f_1, f_2) + H(f_1, \nabla_j^K f_2), \quad j = 1, 2. \quad (1.50)$$

Moreover, if ∂^K is a standard holomorphic structure then the connection ∇^K has constant curvature iff K is a multiple of the identity.

Proof. We write $\partial^K = \partial_{\mathcal{E}} + Z$ with $Z \in \text{End}_{\mathcal{A}_\theta}(\mathcal{E})$ and make the Ansatz

$$\nabla_j^K = \nabla_j + \omega_j, \quad \omega_j \in \mathcal{A}_{\theta'}. \quad (1.51)$$

Then

$$\begin{aligned} \delta_j H(f, g) &= \delta_j \langle Kf, g \rangle_{\mathcal{A}_\theta} = {}_{\mathcal{A}_\theta} \langle \nabla_j(Kf), g \rangle + \langle Kf, \nabla_j g \rangle_{\mathcal{A}_\theta} \\ &= {}_{\mathcal{A}_\theta} \langle (\delta'_j K)f, g \rangle + H(\nabla_j^K f, g) + h(f, \nabla_j^K g) - \langle K\omega_j f, g \rangle_{\mathcal{A}_\theta} - \langle Kf, \omega_j g \rangle_{\mathcal{A}_\theta}. \end{aligned} \quad (1.52)$$

Thus Eq. (1.50) is satisfied iff

$$\delta'_j K = K\omega_j + \omega_j^* K. \quad (1.53)$$

Eq. (1.49) is equivalent to

$$\omega_1 + \bar{\tau}\omega_2 = Z. \quad (1.54)$$

It then follows

$$\begin{aligned} \delta'_\tau K &= K(\omega_1 + \bar{\tau}\omega_2) + (\omega_1^* + \bar{\tau}\omega_2^*)K \\ &= KZ + Z^*K + (\bar{\tau} - \tau)\omega_2^* K. \end{aligned} \quad (1.55)$$

Since $\Im\tau \neq 0$ this determines ω_2 and thus ω_1 uniquely.

Conversely, defining ω_2 by Eq. (1.55) and ω_1 by Eq. (1.54) one easily sees that Eq. (1.53), (1.54) and hence Eq. (1.49), (1.50) are fulfilled.

To prove the last claim we recall, cf. Eq. (1.14), that the standard connection has constant curvature. The holomorphic structure being standard implies that Z is a central element of \mathcal{A}_θ , hence $Z = z \cdot 1$. The curvature of ∇_j^K is given by

$$[\nabla_1^K, \nabla_2^K] = [\nabla_1, \nabla_2] + \delta'_1 \omega_2 + \delta'_2 \omega_1. \quad (1.56)$$

Note that in view of Eq. (1.54) and $Z = zI$ we have $[\omega_1, \omega_2] = 0$. The curvature is constant if and only if

$$\delta'_j(\delta'_1 \omega_2 + \delta'_2 \omega_1) = 0 \quad \text{for } j = 1, 2. \quad (1.57)$$

Multiplying by $\text{rk } \mathcal{E}(g, \theta)^2$ we see in view of Eq. (1.54) that this is equivalent to

$$\delta_j(\delta_1 - \bar{\tau}\delta_2)\omega_2 = 0, \quad \text{for } j = 1, 2. \quad (1.58)$$

Writing $\omega_2 = \sum_{j,l \in \mathbb{Z}} a_{j,l} V_1^k V_2^l$ we find

$$(\delta_1 - \bar{\tau}\delta_2)\omega_2 = 2\pi i \sum_{j,l \in \mathbb{Z}} (j - \bar{\tau}l) a_{j,l} V_1^k V_2^l. \quad (1.59)$$

This is constant iff $(j - \bar{\tau}l)a_{j,l} = 0$ for $(j, l) \neq (0, 0)$. But since $\Im\tau \neq 0$ this is equivalent to ω_2 itself being constant, say $\omega_2 = \lambda I$. By Eq. (1.54) ω_1 is constant, too. Write $\tilde{\lambda} = z + \bar{z} + (\bar{\tau} - \tau)\lambda$. Then in view of Eq. (1.55) we see that K satisfies the equation $\delta'_\tau K = \tilde{\lambda}K$. Expanding K into its Fourier series it then follows that K is a constant multiple of the identity.

For such K we have $\delta'_\tau K = 0$ and Eq. (1.55) implies $\omega_2 \in \mathbb{C} \cdot I$ and thus, by Eq. (1.54), also $\omega_1 \in \mathbb{C} \cdot I$. \square

1.3.2. Curvature of twisted spectral triple. The square of the operator (1.44) is

$$\begin{aligned} \overline{D}_{\mathcal{E}',k}^2 &= -\text{rk}(\mathcal{E}')^2 \begin{pmatrix} \Delta_{\mathcal{E}',k}^+ & 0 \\ 0 & \Delta_{\mathcal{E}',k}^- \end{pmatrix}, \quad \text{where} \\ \Delta_{\mathcal{E}',k}^+ &:= k \partial_{\mathcal{E}'} \partial_{\mathcal{E}'}^* k \quad \text{and} \quad \Delta_{\mathcal{E}',k}^- := \partial_{\mathcal{E}'}^* k^2 \partial_{\mathcal{E}'}. \end{aligned} \quad (1.60)$$

The curvature functionals are defined, as in [CoMo14] (see also Remark 5.4 therein), by means of the constant term in the asymptotic expansion

$$\mathrm{Tr} \left(a e^{-t \Delta_{\mathcal{E}',k}^{\pm}} \right) \sim_{t \searrow 0} \sum_{j=0}^{\infty} a_{2j}(a, \Delta_{\mathcal{E}',k}^{\pm}) t^{j-1}; \quad (1.61)$$

specifically, the *curvature functional* associated to the twisted spectral triple $(A_{\theta}, \mathcal{H}_0(g^{-1}, \theta') \oplus \mathcal{H}_0(g^{-1}, \theta'), \overline{D}_{\mathcal{E}',k})$ is the functional

$$\mathcal{A}_{\theta} \ni a \mapsto \mathcal{R}_{\mathcal{E}',k}^{\pm}(a) := a_2(a, \Delta_{\mathcal{E}',k}^{\pm}). \quad (1.62)$$

The existence of the expansion (1.61) follows from the pseudodifferential calculus for twisted crossed products (Section 2), the resolvent expansion for pseudodifferential multipliers (Section 3, Theorems 3.1, 3.2) and its realization on Heisenberg modules (Section 5, Theorem 5.3). Theorem 5.3 shows moreover that the functional $\mathcal{R}_{\mathcal{E}',k}^{\pm}$ is given by a density $\mathcal{K}_{\mathcal{E}',k}^{\pm} \in \mathcal{A}_{\theta}$ (see Eq. (1.64) below) with respect to the natural trace on $\mathrm{End}_{\mathcal{A}_{\theta'}}(\mathcal{E}') = \mathcal{A}_{\theta}$, namely

$$\varphi_{\mathcal{E}'} := |\mathrm{rk}(\mathcal{E}')| \varphi_0. \quad (1.63)$$

Adjusted by a normalization factor (to be explained below), this density, denoted $\mathcal{K}_{\mathcal{E}',k}^{\pm}$, is given by

$$\mathcal{R}_{\mathcal{E}',k}^{\pm}(a) = \frac{1}{4\pi\Im\tau} \varphi_{\mathcal{E}'}(a \mathcal{K}_{\mathcal{E}',k}^{\pm}) = \frac{|\mathrm{rk}(\mathcal{E}')|}{4\pi\Im\tau} \varphi_0(a \mathcal{K}_{\mathcal{E}',k}^{\pm}), \quad a \in \mathcal{A}_{\theta}. \quad (1.64)$$

Remark 1.11. When comparing with [CoMo14] one has to remember that our basic derivations (1.2) are $2\pi i$ multiples of those used therein. Furthermore, there are different conventions for the modular functions which lead to another overall factor of -2 for the one variable functions and to -4 for two variable functions, cf. Sec. 4.7 below. Multiplying the above factor $\frac{1}{4\pi\Im\tau}$ by $-4\pi^2$ gives precisely the overall factor $-\frac{\pi}{\Im\tau}$ of the expression (3.14) in [CoMo14].

The first main result of the paper computes the precise expression of these densities, thus producing formulas for the *local curvature* of the twisted spectral triple $(A_{\theta}, \mathcal{H}_0(g^{-1}, \theta') \oplus \mathcal{H}_0(g^{-1}, \theta'), \overline{D}_{\mathcal{E}',k})$. With a slight change of notation for the curvature-defining functions introduced in [CoMo14, §3.1], the result can be stated as follows. ¶

Theorem 1.12 (Curvature densities). *Let $h = h^* \in \mathcal{A}_{\theta}^{\mathrm{sa}}$ and $k = e^{\frac{h}{2}}$. Then*

$$a_2(a, \Delta_{\mathcal{E}',k}^{\pm}) \equiv \mathcal{R}_{\mathcal{E}',k}^{\pm}(a) = \frac{|\mathrm{rk}(\mathcal{E}')|}{4\pi\Im\tau} \varphi_0(a \mathcal{K}_{\mathcal{E}',k}^{\pm}), \quad a \in \mathcal{A}_{\theta}. \quad (1.65)$$

where

$$\begin{aligned} \mathcal{K}_{\mathcal{E}',k}^{\pm} &= K_{\pm}(\nabla_h)(\Delta_{\tau}(h)) + H_{\pm}^{\Re}(\nabla_h^1, \nabla_h^2)(\square^{\Re}(h)) \\ &\quad + H_{\pm}^{\Im}(\nabla_h^1, \nabla_h^2)((\square^{\Im}(h)) \mp 2\pi\Im\tau \mu(\mathcal{E}') 1). \end{aligned} \quad (1.66)$$

Here, $\nabla_h = -\mathrm{ad} h$ is the modular derivative, $\Delta_{\tau}h = \delta_{\tau}\delta_{\tau}^*h$ is the Laplacian of the complex structure (cf. Eq. (1.3)),

$$\square^{\Re}(h) = \frac{1}{2}(\partial_{\tau}h \cdot \partial_{\tau}^*h + \partial_{\tau}^*h \cdot \partial_{\tau}h), \quad \square^{\Im}(h) = \frac{1}{2}(\partial_{\tau}h \cdot \partial_{\tau}^*h - \partial_{\tau}^*h \cdot \partial_{\tau}h),$$

¶For the classical case of a Riemann surface, see [Bos87, §1.5], in particular formula (1.5.4).

and ∇_h^i , $i = 1, 2$, signifies that ∇_h is acting on the i -th factor.

K_+ (K_-) equals $K_{0,0}$ ($K_{0,1}$) in Eq. (4.32), $H_+^{\mathfrak{N}}$ ($H_-^{\mathfrak{N}}$) equals $H_{0,0}^{\mathfrak{N}}$ ($H_{0,1}^{\mathfrak{N}}$) in Eq. (4.36) and $H_+^{\mathfrak{J}}$ ($H_-^{\mathfrak{J}}$) equals $H_{0,0}^{\mathfrak{J}}$ ($H_{0,1}^{\mathfrak{J}}$) in Eq. (4.34).

Proof. With the modular operator $\Delta := k^{-2} \cdot k^2 = e^{\nabla_h}$ we first note that $\Delta_{\mathcal{E}',k}^+ = k \partial_{\mathcal{E}'} \partial_{\mathcal{E}',k}^* = \Delta^{\frac{1}{2}} (k^2 \partial_{\mathcal{E}'} \partial_{\mathcal{E}',k}^*)$, hence (see 4.5.1)

$$\mathrm{Tr}(a e^{-t \Delta_{\mathcal{E}',k}^+}) = \mathrm{Tr}(k a k^{-1} e^{-t k^2 \partial_{\mathcal{E}'} \partial_{\mathcal{E}',k}^*}).$$

We therefore apply Theorem 5.3 to the operator $k^2 \partial_{\mathcal{E}'} \partial_{\mathcal{E}',k}^*$. Its symbol equals (cf. Sec. 2.3.2 and Sec. 3.1) $k^2 |\eta|^2 + \frac{1}{2} k^2 c_\tau$, where $c_\tau = [\partial_{\mathcal{E}'}, \partial_{\mathcal{E}',k}^*] = 4\pi \mu(\mathcal{E}') \mathfrak{I} \tau$ is the structure constant (cf. Eq. (2.33)). According to Theorem 5.3 and Corollary 3.3 the constant term $\frac{1}{2} k^2 c_\tau$ in the symbol contributes to $a_2(a, \Delta_{\mathcal{E}',k}^+)$ the summand

$$\frac{N(\pi)}{4\pi \mathfrak{I} \tau} \varphi_0(k a k^{-1} (-k^{-2} \frac{1}{2} k^2 c_\tau)) = -\frac{|\mathrm{rk}(\mathcal{E}')| \mu(\mathcal{E}')}{2} \varphi_0(a) = -\frac{\mu(\mathcal{E}')}{2} \varphi_{\mathcal{E}'}(a).$$

Note that for the module \mathcal{E}' the universal constant $N(\pi)$ (cf. Theorem 5.2) equals $|\mathrm{rk}(\mathcal{E}')|$. The first large summand on the right of Eq. (1.66) follows from Corollary 3.3 as well. The normalization constant in front of the last expression in Eq. (1.65) is (cf. Theorems 5.2, 5.3) $\frac{N(\pi)}{4\pi \mathfrak{I} \tau} = \frac{|\mathrm{rk}(\mathcal{E}')|}{4\pi \mathfrak{I} \tau}$. \square

Remark 1.13. An equivalent formulation of Eq. (1.66) is

$$\mathcal{K}_{\mathcal{E}',k}^\pm = \mathcal{K}_k^\pm \mp 2\pi \mathfrak{I} \tau \mu(\mathcal{E}') 1, \quad (1.67)$$

where \mathcal{K}_k^\pm is the intrinsic scalar curvature of T_θ^2 (cf. [CoMo14, Sec. 4]) equipped with the conformal metric of Weyl factor $k = e^{\frac{h}{2}}$. Note though that due to the change of conventions (cf. Remark 1.11) the latter is a $-4\pi^2$ multiple of that defined in [CoMo14].

Corollary 1.14 (Riemann-Roch density). *The Riemann-Roch density $\mathcal{K}_{\mathcal{E}',k}^\gamma = \mathcal{K}_{\mathcal{E}',k}^+ - \mathcal{K}_{\mathcal{E}',k}^-$ is given by the formula*

$$\begin{aligned} \mathcal{K}_{\mathcal{E}',k}^\gamma &= K_\gamma(\nabla_h)(\Delta_\tau(h)) + H_\gamma^{\mathfrak{N}}(\nabla_h^1, \nabla_h^2)(\square^{\mathfrak{N}}(h)) \\ &\quad + H_\gamma^{\mathfrak{J}}(\nabla_h^1, \nabla_h^2)((\square^{\mathfrak{J}}(h)) - 4\pi \mathfrak{I} \tau \mu(\mathcal{E}') 1). \end{aligned} \quad (1.68)$$

The expressions of K_γ , $H_\gamma^{\mathfrak{N}}$, $H_\gamma^{\mathfrak{J}}$ are explicitly given in Eq. (4.31), (4.35), (4.33).

The case $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, i.e., of the trivial “line bundle” with a standard holomorphic structure was treated in [KhMo14].

1.4. Ray-Singer determinant functional and curvature.

1.4.1. Ray-Singer determinant for Heisenberg bimodules. In the preceding section we started by performing a conformal change of metric in the base. However, after passing to the transposed spectral triple, the initial algebra \mathcal{A}_θ becomes the algebra of endomorphisms $\mathrm{End}_{\mathcal{A}_\theta}(\mathcal{E}')$. In the new picture, the conformal change on the base amounts to a completely general change of Hermitian metric on the right

$\mathcal{A}_{\theta'}$ -module \mathcal{E}' . Indeed, any Hermitian structure on \mathcal{E}' , viewed as a Hilbert C^* -module over $\mathcal{A}_{\theta'}$, can be obtained from the standard one by composition with a positive $\mathcal{A}_{\theta'}$ -linear endomorphism, *i.e.*, a positive element in $\mathcal{A}_{\theta'}$.

From now on we assume $\mu(\mathcal{E}') > 0$. By Proposition 1.5 (1) this implies

$$\text{Ker } \Delta_{\mathcal{E}'}^+ = \text{Ker } \partial_{\mathcal{E}'}^* = 0.$$

It then follows that $\text{Ker } \Delta_{\mathcal{E}',k}^+ = 0$ for any invertible $k = e^{\frac{h}{2}} \in \mathcal{A}_{\theta'}^{\text{sa}}$. Once the above choice of the sign of $\mu(\mathcal{E}')$ was made, we will only deal with the Hodge-Laplace operator on “functions”. Accordingly, we shall routinely omit the superscript $+$ when referring to $\Delta_{\mathcal{E}',k}^+$ as well as to the rest of symbols affected by the same notation.

Having no zero modes, the zeta function of $\Delta_{\mathcal{E}',k}$ is

$$\zeta_{\Delta_{\mathcal{E}',k}}(z) = \text{Tr}(\Delta_{\mathcal{E}',k}^{-z}) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \text{Tr}(e^{-t\Delta_{\mathcal{E}',k}}) dt.$$

Its value at 0 is independent of the Weyl factor $k \in \mathcal{A}_{\theta'}$, as shown by the proof of [CoMo14, Theorem 2.2]. By Proposition 1.5 (2) this value is

$$\zeta_{\Delta_{\mathcal{E}',k}}(0) = \zeta_{\Delta_{\mathcal{E}'}}(0) = -\frac{|\deg(\mathcal{E}')|}{2}. \quad (1.69)$$

On the other hand, the corresponding Ray-Singer determinant varies and gives rise to the functional

$$\mathcal{A}_{\theta'}^{\text{sa}} \ni h^* = h \mapsto \text{RS}_{\mathcal{E}',k} := \log \text{Det}(\Delta_{\mathcal{E}',k}) := -\zeta'_{\Delta_{\mathcal{E}',k}}(0). \quad (1.70)$$

To obtain the variation formula for this functional, one proceeds as in [CoMo14, §4.1]. First, by [CoMo14, Eq. (4.1)]

$$-\frac{d}{ds} \zeta'_{\Delta_{\mathcal{E}',k^s}}(0) = \zeta_{\Delta_{\mathcal{E}',k^s}}(h, 0). \quad (1.71)$$

Applying Theorem 1.12 to the right hand side one obtains

$$\begin{aligned} -\frac{d}{ds} \zeta'_{\Delta_{\mathcal{E}',k^s}}(0) = & \quad (1.72) \\ & \frac{|\text{rk}(\mathcal{E}')|}{4\pi\mathfrak{I}\tau} \varphi_0 \left(h \left(sK_+(s\nabla_h)(\Delta(h)) + s^2 H_+^{\mathfrak{R}}(s\nabla_h^1, s\nabla_h^2)(\square^{\mathfrak{R}}(h)) \right) \right) \\ & - \frac{1}{2} |\text{rk}(\mathcal{E}')| \mu(\mathcal{E}') \varphi_0(h). \end{aligned}$$

Notice that

$$|\text{rk}(\mathcal{E}')| \mu(\mathcal{E}') = \text{sgn}(\text{rk}(\mathcal{E}')) \deg(\mathcal{E}') = |\deg(\mathcal{E}')|, \quad (1.73)$$

since

$$\text{sgn}(\text{rk}(\mathcal{E}')) \text{sgn}(\deg(\mathcal{E}')) = \text{sgn } \mu(\mathcal{E}') = 1.$$

By integrating Eq. (1.72) and using the observation (1.73) one obtains the variation formula

$$\begin{aligned} \log \text{Det}(\Delta_{\mathcal{E}',k}) = \log \text{Det}(\Delta_{\mathcal{E}'}) - \frac{1}{2} |\deg(\mathcal{E}')| \varphi_0(h) \\ + \frac{|\text{rk}(\mathcal{E}')|}{4\pi\mathfrak{I}\tau} \int_0^1 \varphi_0 \left[h \left(sK_+(s\nabla_h)(\Delta(h)) + s^2 H_+^{\mathfrak{R}}(s\nabla_h^1, s\nabla_h^2)(\square^{\mathfrak{R}}(h)) \right) \right] ds. \end{aligned} \quad (1.74)$$

The integrand above is carefully evaluated in [CoMo14, Lemmas 4.2 - 4.4]. As a result, the above identity becomes

$$\begin{aligned} \log \text{Det}(\Delta_{\mathcal{E}',k}) &= \log \text{Det}(\Delta_{\mathcal{E}'}) - \frac{1}{2} |\deg(\mathcal{E}')| \varphi_0(h) \\ &\quad - \frac{|\text{rk}(\mathcal{E}')|}{16\pi\mathfrak{I}\tau} \left(\frac{1}{3} \varphi_0(h\Delta h) + \varphi_0 \left(K_2(\nabla_h^1)(\square^{\mathfrak{R}}(h)) \right) \right), \end{aligned} \quad (1.75)$$

with the function K_2 given by

$$K_2(2s) = \frac{1}{3} - \left(\frac{\coth(s)}{s} - \frac{1}{s^2} \right), \quad s \in \mathbb{R}. \quad (1.76)$$

Theorem 1.15 (Determinant formula). *The exact formula for the Ray-Singer determinant corresponding to the change of Hermitian metric on the Heisenberg right $\mathcal{A}_{\theta'}$ -module \mathcal{E}' by the factor $k = e^{\frac{1}{2}h} \in \mathcal{A}_{\theta}^{\text{sa}}$ is*

$$\begin{aligned} \log \text{Det}(\Delta_{\mathcal{E}',k}) &= \frac{1}{2} |\deg(\mathcal{E}')| \log(2|\mu(\mathcal{E}')|\mathfrak{I}(\bar{\tau})) - \frac{1}{2} |\deg(\mathcal{E}')| \varphi_0(h) \\ &\quad - \frac{|\text{rk}(\mathcal{E}')|}{16\pi\mathfrak{I}\tau} \left(\frac{1}{3} \varphi_0(h\Delta h) + \varphi_0 \left(K_2(\nabla_h^1)(\square^{\mathfrak{R}}(h)) \right) \right). \end{aligned} \quad (1.77)$$

Proof. The stated formula follows from Eq. (1.75), using Eq. (1.32) to express the value of the log-determinant for the constant curvature metric. \square

1.4.2. Extremal of Ray-Singer determinant functional. We take the point of view the Ray-Singer determinant is a functional on the positive cone of Hermitian metrics on the $\mathcal{A}_{\theta'}$ -module \mathcal{E}' . Actually, we shall regard it as a functional on the space $\mathcal{A}_{\theta}^{\text{sa}}$, by composing with the map $\mathcal{A}_{\theta}^{\text{sa}} \ni h \mapsto k = e^{\frac{1}{2}h}$.

Under the rescaling $h \mapsto h + \epsilon$ one has

$$\begin{aligned} \zeta'_{\Delta_{\mathcal{E}',k}e^{\epsilon/2}}(0) &= \frac{d}{dz} \Big|_{z=0} (e^{-\epsilon z} \zeta_{\Delta_{\mathcal{E}',k}}(z)) = \zeta'_{\Delta_{\mathcal{E}',k}}(0) - \epsilon \zeta_{\Delta_{\mathcal{E}',k}}(0) \\ &= \zeta'_{\Delta_{\mathcal{E}',k}}(0) + \epsilon \frac{|\deg(\mathcal{E}')|}{2}. \end{aligned}$$

It follows that the modified functional (analogous to that in [OPS88])

$$F_{\mathcal{E}'}(h) = -\log \text{Det}(\Delta_{\mathcal{E}',k}) - \frac{1}{2} |\deg(\mathcal{E}')| \varphi_0(h) \quad (1.78)$$

is scale invariant. The identity (1.75) yields

$$\begin{aligned} F_{\mathcal{E}'}(h) &= -\frac{1}{2} |\deg(\mathcal{E}')| \varphi_0(h) - \log \text{Det}(\Delta_{\mathcal{E}'}) + \frac{1}{2} |\deg(\mathcal{E}')| \varphi_0(h) \\ &\quad + \frac{|\text{rk}(\mathcal{E}')|}{16\pi\mathfrak{I}\tau} \left(\frac{1}{3} \varphi_0(h\Delta h) + \varphi_0 \left(K_2(\nabla_h^1)(\square^{\mathfrak{R}}(h)) \right) \right), \end{aligned}$$

which after cancelation becomes

$$\begin{aligned} F_{\mathcal{E}'}(h) &= -\log \text{Det}(\Delta_{\mathcal{E}'}) \\ &\quad + \frac{|\text{rk}(\mathcal{E}')|}{16\pi\mathfrak{I}\tau} \left(\frac{1}{3} \varphi_0(h\Delta h) + \varphi_0 \left(K_2(\nabla_h^1)(\square^{\mathfrak{R}}(h)) \right) \right). \end{aligned} \quad (1.79)$$

Theorem 1.16 (Extremal value). *The scale invariant Ray-Singer determinant $k^2 \mapsto F_{\mathcal{E}'}(\log(k^2))$, viewed as a functional on the (positive cone of) metrics on the Heisenberg left \mathcal{A}_θ -module \mathcal{E}' , attains its minimum only at the metric whose corresponding unique metric connection compatible with the holomorphic structure has constant curvature.*

Proof. This follows from Eq. (1.79) and the crucial positivity result established in the proof of [CoMo14, Theorem 4.6]

$$\frac{1}{3}\varphi_0(h\Delta h) + \varphi_0\left(K_2(\nabla_h^1)(\square^{\mathfrak{R}}(h))\right) \geq 0,$$

for all $h^* = h \in \mathcal{A}_\theta$, with the equality holding only for $h = 0$. \square

Recalling that by Prop. 1.10 the standard holomorphic structure and the Hermitian metric uniquely determines the metric connection, the above result places the (spectral) Ray-Singer determinant functional in a role similar to that of the (local) Yang-Mills functional, albeit restricted to connections compatible with the holomorphic structure.

1.4.3. Gradient of Ray-Singer determinant. In this section we recover the intrinsic curvature of the conformal metric on the base from the gradient of the Ray-Singer determinant functional associated to a Heisenberg bimodule.

We define the gradient of the functional $F_{\mathcal{E}'}$ by the equation

$$\begin{aligned} \langle \text{grad}_h F_{\mathcal{E}'}, a \rangle_{\mathcal{E}'} &\equiv \frac{|\text{rk}(\mathcal{E}')|}{4\pi\mathfrak{I}\tau} \varphi_0(a \cdot \text{grad}_h F_{\mathcal{E}'}) \\ &:= \frac{d}{d\epsilon} \Big|_{\epsilon=0} F_{\mathcal{E}'}(h + \epsilon a), \quad a = a^* \in \mathcal{A}_\theta^\infty. \end{aligned} \quad (1.80)$$

Its explicit expression can be computed as in [CoMo14, §4.2]. Indeed, in the absence of zero-modes, the calculation in the proof of [CoMo14, Thm. 4.8] gives for the gradient of the Ray-Singer functional

$$\text{RS}_{\mathcal{E}'}(h) := -\zeta'_{\Delta_{\mathcal{E}',k}}(0) \quad (1.81)$$

the following formula

$$\frac{d}{d\epsilon} \Big|_{\epsilon=0} \text{RS}_{\mathcal{E}'}(h + \epsilon a) = \frac{1}{2} a_2 \left(\int_{-1}^1 e^{\frac{uh}{2}} a e^{-\frac{uh}{2}} du, \Delta_{\mathcal{E}',k} \right), \quad a \in \mathcal{A}_\theta. \quad (1.82)$$

From the definition (1.78) it follows that

$$\frac{d}{d\epsilon} \Big|_{\epsilon=0} F(h + \epsilon a) = -\frac{1}{2} a_2 \left(\int_{-1}^1 e^{\frac{uh}{2}} a e^{-\frac{uh}{2}} du, \Delta_{\mathcal{E}',k} \right) - \frac{1}{2} |\text{deg}(\mathcal{E}')| \varphi_0(a).$$

Using Eq. (1.73) and the above definition of the gradient, this yields

$$\begin{aligned} &\frac{|\text{rk}(\mathcal{E}')|}{4\pi\mathfrak{I}\tau} \varphi_0(a \cdot \text{grad}_h F_{\mathcal{E}'}) \\ &= -\frac{1}{2} a_2 \left(\int_{-1}^1 e^{\frac{uh}{2}} a e^{-\frac{uh}{2}} du, \Delta_{\mathcal{E}',k} \right) - \frac{1}{2} \mu(\mathcal{E}') |\text{rk}(\mathcal{E}')| \varphi_0(a). \end{aligned} \quad (1.83)$$

By Theorem 1.12, Eq. (1.65), one has

$$\begin{aligned} a_2 \left(\int_{-1}^1 e^{\frac{u\hbar}{2}} a e^{-\frac{u\hbar}{2}} du, \triangle_{\mathcal{E}',k} \right) &= \frac{|\mathrm{rk}(\mathcal{E}')|}{4\pi\Im\tau} \varphi_0 \left(a \cdot \int_{-1}^1 e^{\frac{u\hbar}{2}} \mathcal{K}_{\mathcal{E}',h} e^{-\frac{u\hbar}{2}} du \right) \\ &= \frac{|\mathrm{rk}(\mathcal{E}')|}{4\pi\Im\tau} \varphi_0 \left(a \cdot \tilde{\mathcal{K}}_{\mathcal{E}',k} \right), \end{aligned}$$

where we have denoted

$$\tilde{\mathcal{K}}_{\mathcal{E}',k} := \int_{-1}^1 e^{\frac{u\hbar}{2}} \mathcal{K}_{\mathcal{E}',k}^+ e^{-\frac{u\hbar}{2}} du \equiv 2 \frac{\sinh(\nabla/2)}{\nabla/2} (\mathcal{K}_{\mathcal{E}',k}^+). \quad (1.84)$$

Theorem 1.17 (Gradient formula).

$$\mathrm{grad}_h F_{\mathcal{E}'} = -\frac{1}{2} \tilde{\mathcal{K}}_k. \quad (1.85)$$

Proof. With the above notation the identity (1.83) becomes

$$\frac{|\mathrm{rk}(\mathcal{E}')|}{4\pi\Im\tau} \varphi_0(a \, \mathrm{grad}_h F_{\mathcal{E}'}) = -\frac{|\mathrm{rk}(\mathcal{E}')|}{8\pi\Im\tau} \varphi_0 \left(a \cdot \tilde{\mathcal{K}}_{\mathcal{E}',k} \right) - \frac{1}{2} \mu(\mathcal{E}') |\mathrm{rk}(\mathcal{E}')| \varphi_0(a).$$

Hence

$$\mathrm{grad}_h F_{\mathcal{E}'} = -\frac{1}{2} \tilde{\mathcal{K}}_{\mathcal{E}',k} - 2\pi\Im\tau \mu(\mathcal{E}') 1.$$

By Eq. (1.67), the right hand side is equal to

$$-\frac{1}{2} \left(\tilde{\mathcal{K}}_k - 2\pi\Im\tau \mu(\mathcal{E}') 2 \frac{\sinh(\nabla/2)}{\nabla/2} (1) \right) - 2\pi\Im\tau \mu(\mathcal{E}') 1 = -\frac{1}{2} \tilde{\mathcal{K}}_k. \quad \square$$

Now Theorem 4.8 in [CoMo14] states that for the case of trivial coefficients,

$$\mathrm{grad}_h F = -\frac{1}{2} \tilde{\mathcal{K}}_k.$$

Thus, the above result can be rephrased as follows.

Corollary 1.18 (Morita invariance).

$$\mathrm{grad}_h F_{\mathcal{E}'} = \mathrm{grad}_h F.$$

If one adopts the point of view that the *Gaussian curvature* is, by definition, the gradient of the Ray-Singer determinant functional, the above identity establishes both its Morita invariance and its independence of the Spin^c -structure. Indeed, the *non-twisted* spectral triples in Proposition 1.7 or Proposition 1.9 define non-commutative metric structures on $A_{\theta'}$ in the sense of Connes' foundational axioms [Con96]. These are obtained by coupling the standard Dirac spectral triple with arbitrary Hermitian metrics on the “bundles” \mathcal{E}' , which are automatically as in Eq. (1.47), for some positive, invertible $k \in \mathrm{End}_{A_{\theta'}}(\mathcal{E}') = A_{\theta}$. (In the case of commutative tori, by Connes' reconstruction theorem [Con13], they actually exhaust all the Spin^c -Dirac operators on such tori.) What we have proved is that the Gaussian curvature of such a metric on the noncommutative torus $A_{\theta'}$ coincides with the intrinsic metric on A_{θ} corresponding to the same $k \in A_{\theta}$, and in particular is independent on the “ Spin^c -structure” \mathcal{E}' .

2. PSEUDODIFFERENTIAL MULTIPLIERS AND SYMBOL CALCULUS

2.1. Twisted crossed product. To establish an appropriate pseudodifferential calculus on Heisenberg modules we will need an extension to twisted crossed products of Connes' [CON80] and Baa's [BAA88A, BAA88B] pseudodifferential calculus for C^* -dynamical systems. We therefore consider a C^* -dynamical system $(\mathcal{A}, \mathbb{R}^n, \alpha)$ with unital \mathcal{A} . That is α is a strongly continuous action of the additive group \mathbb{R}^n as automorphisms on the C^* -algebra \mathcal{A} . Furthermore, let $B = (b_{kl})_{k,l=1}^n$ be a skew-symmetric real $n \times n$ -matrix. Put

$$e(x, y) := e^{i\sigma(x, y)} = e^{i\langle Bx, y \rangle}, \quad \sigma(x, y) := \langle Bx, y \rangle. \quad (2.1)$$

The skew-symmetry of B implies $e(x, x) = 1$ which will be used silently many times. Our main examples are the C^* -dynamical systems associated to projective representations of \mathbb{R}^2 on the modules $\mathcal{E}(g, \theta)$ as described in Section 1.2.1 and in the Appendix A. The bilinear form σ in Eq. (2.1) corresponds to $\mu(g, \theta)\sigma$ in Eq. (1.16), cf. also Eq. (1.19), with

$$B = \pi \mu(g, \theta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b_{21} = -b_{12} = \pi \mu(g, \theta). \quad (2.2)$$

For $a \in \mathcal{A}^\infty$ and a multiindex $\gamma \in \mathbb{Z}_+^n$ we put as usual

$$\delta^\gamma a := i^{|\gamma|} \partial_x^\gamma \big|_{x=0} \alpha_x(a) = i^{-|\gamma|} \partial_x^\gamma \big|_{x=0} \alpha_{-x}(a). \quad (2.3)$$

δ^γ plays the role of the partial derivative $i^{-|\gamma|} \frac{\partial^\gamma}{\partial x^\gamma}$. \mathcal{A}^∞ together with the seminorms $\|a\|_\gamma := \|\delta^\gamma a\|$ is a Fréchet space. The Schwartz space $\mathcal{S}(\mathbb{R}^n, \mathcal{A}^\infty)$ is then defined as usual or as the projective tensor product of the (scalar) Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and \mathcal{A}^∞ .

For the following see also Appendix A.2. $\mathcal{S}(\mathbb{R}^n, \mathcal{A}^\infty)$ is a pre- C^* -module with inner product

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)^* g(x) dx. \quad (2.4)$$

Put

$$(af)(x) = \alpha_{-x}(a)f(x), \quad a \in \mathcal{A}^\infty; \quad (2.5)$$

$$(U_y f)(x) = e(x, -y)f(x - y). \quad (2.6)$$

$U_y, y \in \mathbb{R}^n$, is a projective family of unitaries which implements the group of automorphisms $\alpha_y, y \in \mathbb{R}^n$:

$$U_x^* = U_{-x}, \quad U_x U_y = e(x, y) U_{x+y}, \quad x, y \in \mathbb{R}^n, \quad (2.7)$$

$$U_x a U_{-x} = \alpha_x(a), \quad a \in \mathcal{A}^\infty. \quad (2.8)$$

By associating to $f \in \mathcal{S}(\mathbb{R}^n, \mathcal{A}^\infty)$ the multiplier $M_f = \int_{\mathbb{R}^n} f(x) U_x dx$ the space $\mathcal{S}(\mathbb{R}^n, \mathcal{A}^\infty)$ becomes a $*$ -algebra. Explicitly, $M_f \circ M_g = M_{f*g}$ and $M_f^* = M_{f^*}$, where

$$\begin{aligned} f^*(x) &= \alpha_x(f(-x)^*), \\ (f * g)(x) &= \int_{\mathbb{R}^n} f(y) \alpha_y(g(x - y)) e(y, x) dy. \end{aligned} \quad (2.9)$$

Note that the formula for f^* is the same as in the untwisted case. If we want to emphasize the dependence of the product on the twisting σ we write $*_\sigma$.

2.1.1. *Dual Trace on $\mathcal{S}(\mathbb{R}^n, \mathcal{A}^\infty)$.* If ψ is an α -invariant (finite) trace on \mathcal{A} then the dual trace $\widehat{\psi}$ on $\mathcal{S}(\mathbb{R}^n, \mathcal{A}^\infty)$ is given by

$$\widehat{\psi}(f) = \psi(f(0)) = \int_{\mathbb{R}^n} \psi(\widehat{f}(\xi)) \, d\xi, \quad d\xi = (2\pi)^{-n} d\xi. \quad (2.10)$$

Note that $d\xi$ is the Plancherel measure of the dual group $(\mathbb{R}^n)^\wedge$ w.r.t. the duality pairing $(x, \xi) \mapsto e^{i\langle x, \xi \rangle}$.

2.2. **Pseudodifferential multipliers.** The symbol spaces (of Hörmander type $(1, 0)$) $S^m(\mathbb{R}^n, \mathcal{A}^\infty)$ are defined as in [CON80, BAA88A, BAA88B]. We will also consider classical (1-step polyhomogeneous) symbols $f \in CS^m(\mathbb{R}^n, \mathcal{A}^\infty)$ which have an asymptotic expansion

$$f \sim \sum_{j=0}^{\infty} f_{m-j}$$

with $f_{m-j}(\lambda\xi) = \lambda^{m-j} \cdot f_{m-j}(\xi)$, $|\xi| \geq 1, \lambda \geq 1$. Additionally, for the resolvent expansion we will need to consider parameter dependent symbols $S^m(\mathbb{R}^n \times \Gamma, \mathcal{A}^\infty)$ and corresponding pseudodifferential multipliers. This calculus is a little more subtle than the name suggests. The parameter dependent class is more restricted than just having operator families depending on a parameter. It is designed to analyze the resolvent expansion. Here Γ is an open conic subset of the complex plane and the parameter $\lambda \in \Gamma$ is (for the resolvents of second order operators) treated as a variable of degree 2. This is in complete analogy to [SHU01, §9]. For a survey, see [LES10, §3].

To motivate the definition of pseudodifferential multipliers we calculate for Schwartz functions $f, u \in \mathcal{S}(\mathbb{R}^n, \mathcal{A}^\infty)$, with f^\vee denoting the inverse Fourier transform of f ,

$$(M_{f^\vee} u)(x) := \left(\int_{\mathbb{R}^n} f^\vee(y) U_y u \, dy \right)(x) \quad (2.11)$$

$$\begin{aligned} &= \int_{\mathbb{R}^n} \alpha_{-x}(f^\vee(y)) u(x-y) e(x, -y) \, dy = \int_{\mathbb{R}^n} \alpha_{-x}(f^\vee(x-y)) u(y) e(x, y) \, dy \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y, \xi - Bx \rangle} \alpha_{-x}(f(\xi)) u(y) \, dy \, d\xi \end{aligned} \quad (2.12)$$

$$= \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \alpha_{-x}(f(\xi)) \widehat{u}(\xi - Bx) \, d\xi = \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \alpha_{-x}(f(\xi + Bx)) \widehat{u}(\xi) \, d\xi, \quad (2.13)$$

$$=: (P_f u)(x). \quad (2.14)$$

The last three integrals exist (Eq. (2.12) as iterated integral) also if $f \in S^m(\mathbb{R}^n, \mathcal{A}^\infty)$ is a symbol of order m . We call the so defined multiplier P_f a *(twisted) pseudodifferential multiplier with symbol f* .

As in the untwisted case the symbol can be recovered from the multiplier. Namely, for $t \in \mathbb{R}, b \in \mathcal{A}^\infty$ we have

$$e^{-i\langle t, x \rangle} (P_f e^{i\langle t, \cdot \rangle} b)(x) = \alpha_{-x}(f(t + Bx)) b. \quad (2.15)$$

$f(t)$ is recovered by setting $x = 0$ and $b = 1_{\mathcal{A}}$.

Definition 2.1. By $L_\sigma^m(\mathbb{R}^n, \mathcal{A}^\infty)$ we denote the space of pseudodifferential multipliers of the form Eq. (2.12), (2.13). $f =: \sigma(P_f)$ is called the symbol of P_f . The spaces of classical symbols and multipliers are denoted by a decorator C in front, i.e., $CS^m(\mathbb{R}^n, \mathcal{A}^\infty)$ resp. $CL_\sigma^m(\mathbb{R}^n, \mathcal{A}^\infty)$.

The space of twisted (classical) pseudodifferential multipliers is a $*$ -algebra:

Theorem 2.2. For symbols $f \in S^m(\mathbb{R}^n, \mathcal{A}^\infty)$, $g \in S^{m'}(\mathbb{R}^n, \mathcal{A}^\infty)$ the composition $P_f \circ P_g$ is a pseudodifferential multiplier with symbol $h \in S^{m+m'}(\mathbb{R}^n, \mathcal{A}^\infty)$. h has the following asymptotic expansion

$$h(t) \sim \sum_\gamma \frac{i^{-|\gamma|}}{\gamma!} (\partial^\gamma f)(t) \partial_y^\gamma|_{y=0} \left(\alpha_{-y}(g(t + By)) \right). \quad (2.16)$$

Furthermore, P_f^* is a pseudodifferential multiplier with symbol

$$\sigma(P_f^*) \sim \sum_\gamma \frac{1}{\gamma!} \partial_t^\gamma \delta^\gamma f(t)^*. \quad (2.17)$$

Remark 2.3. We compare these formulas with the corresponding formulas in the untwisted case, cf. [Con80]:

The formula for the symbol of the adjoint is unchanged. For the product denote by $h_{\text{untwisted}}$ the symbol of the composition $P_f \circ P_g$ in the untwisted ($B = 0$) calculus. Then

$$h_{\text{untwisted}}(t) \sim \sum_\gamma \frac{i^{-|\gamma|}}{\gamma!} (\partial^\gamma f)(t) \partial_y^\gamma|_{y=0} \alpha_{-y}(g(t)). \quad (2.18)$$

Expanding $\partial_y^\gamma|_{y=0}$ in Eq. (2.16) and counting orders we see that up to terms of order $m + m' - 3$ we have

$$h(t) = h_{\text{untwisted}}(t) + \frac{1}{i} \sum_{j,k=1}^n (\partial_j f)(t) b_{kj} (\partial_k g)(t) \mod S^{m+m'-3}(\mathbb{R}^n, \mathcal{A}^\infty). \quad (2.19)$$

Remark 2.4. The dual trace (Sec. 2.1.1) gives rise to a *natural* trace on the algebra of pseudodifferential multipliers of order $< -n$. Namely, if ψ is an α -invariant (finite) trace on \mathcal{A} then for $f \in S^m(\mathbb{R}^n, \mathcal{A}^\infty)$, $m < -n$, put (cf. Eq. (2.11) – (2.14))

$$\text{Tr}_\psi(P_f) = \widehat{\psi}(f^\vee) = \int_{\mathbb{R}^n} \psi(f(\xi)) d\xi. \quad (2.20)$$

Then Tr_ψ is a trace on $\cup_{m < -n} L_\sigma^m(\mathbb{R}^n, \mathcal{A}^\infty)$. It is important to understand that this trace in general differs from the Hilbert space trace induced by the action on a Heisenberg module. This issue is addressed in Sec. 5.

Proof. From Eq. (2.15) we infer $(P_g e^{i\langle t, \cdot \rangle} 1_{\mathcal{A}})(x) = \alpha_{-x}(g(t + Bx)) e^{i\langle t, x \rangle}$. Thus

$$\begin{aligned}
h(t) &= (P_f P_g e^{i\langle t, \cdot \rangle} 1_A)(0) \\
&= \left(\int_{\mathbb{R}^n} f^\vee(y) U_y \alpha_{-y}(g(t + B \cdot)) e^{i\langle t, \cdot \rangle} dy \right)(0) \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle y, \xi - t \rangle} f(\xi) \alpha_y(g(t - By)) dy d\xi \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i\langle y, \xi \rangle} f(\xi + t) \alpha_{-y}(g(t + By)) dy d\xi.
\end{aligned} \tag{2.21}$$

The usual stationary phase argument implies that the symbol expansion is obtained by Taylor expanding $f(t + \xi) \sim \sum_\gamma \frac{\partial^\gamma f(t)}{\gamma!} \xi^\gamma$, replacing $\xi^\gamma e^{-i\langle y, \xi \rangle}$ by $i^{|\gamma|} \partial_y^\gamma e^{-i\langle y, \xi \rangle}$ and then integrating by parts. Then by the Fourier inversion Theorem the summand of index γ becomes

$$\begin{aligned}
\frac{1}{\gamma!} (\partial^\gamma f)(t) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i\langle y, \xi \rangle} i^{-|\gamma|} \partial_y^\gamma (\alpha_{-y}(g(t + By))) dy d\xi \\
= \frac{i^{-|\gamma|}}{\gamma!} (\partial^\gamma f)(t) \partial_y^\gamma \Big|_{y=0} (\alpha_{-y}(g(t))), \tag{2.22}
\end{aligned}$$

and the product formula is proved.

For the adjoint note first that $P_f^* = M_{f^\vee}^* = M_{f^\vee}^* = P_{(f^\vee)^*} =: P_h$. By Eq. (2.9) we find

$$f^{\vee*}(x) = \alpha_x(f^\vee(-x)^*) = \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \alpha_x(f(\xi)^*) d\xi. \tag{2.23}$$

Thus

$$\begin{aligned}
h(t) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x, \xi - t \rangle} \alpha_x(f(\xi)^*) d\xi dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \alpha_x(f(t + \xi)^*) d\xi dx.
\end{aligned} \tag{2.24}$$

As before we expand $f(t + \xi) \sim \sum_\gamma \frac{\partial^\gamma f(t)}{\gamma!} \xi^\gamma$, replace $\xi^\gamma e^{i\langle x, \xi \rangle}$ by $i^{-|\gamma|} \partial_x^\gamma e^{i\langle x, \xi \rangle}$ and then integrate by parts. The Fourier inversion Theorem then gives the result. \square

2.3. Differential multipliers. Among pseudodifferential multipliers the class of differential multipliers is characterized by having symbols which are polynomial in ξ . Thus

Definition 2.5. $P_f \in L_\beta^m(\mathbb{R}^n, \mathcal{A}^\infty)$ is called a differential multiplier of order m if $f \in \mathcal{A}^\infty[\xi_1, \dots, \xi_n]$ is a polynomial of degree at most m , i.e.,

$$f(\xi) = \sum_{|\gamma| \leq m} a_\gamma \xi^\gamma; \quad a_\gamma \in \mathcal{A}^\infty. \tag{2.25}$$

Here the sum runs over all multiindices $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \leq m$. We denote the space of all differential multipliers by $\text{Diff}_\sigma^m(\mathbb{R}^n, \mathcal{A}^\infty)$. Clearly, differential multipliers are classical pseudodifferential multipliers.

Recall that in the ordinary pseudodifferential calculus the symbol of the basic derivatives $i^{-|\gamma|} \partial_x^\gamma$ is given by ξ^γ . Therefore, for a multiindex γ we put $\partial^\gamma := P_{\xi^\gamma}$.

Proposition 2.6. For $u \in \mathcal{S}(\mathbb{R}^n, \mathcal{A}^\infty)$ we have

$$\begin{aligned} (\partial^\gamma u)(x) &= (P_{\xi^\gamma} u)(x) = i^{|\gamma|} \partial_y^\gamma \big|_{y=0} U_y u(x) \\ &= i^{-|\gamma|} \partial_y^\gamma \big|_{y=0} (e(x, y) u(x + y)). \end{aligned}$$

Proof. We use the definition Eq. (2.13) and find

$$\begin{aligned} (\partial^\gamma u)(x) &= \int_{\mathbb{R}^n} e^{i\langle x, y \rangle} (\xi + Bx)^\gamma \hat{u}(\xi) d\xi \\ &= i^{-|\gamma|} \partial_y^\gamma \big|_{y=0} \int_{\mathbb{R}^n} e^{i\langle x+y, \xi+Bx \rangle} \hat{u}(\xi) d\xi \\ &= i^{-|\gamma|} \partial_y^\gamma \big|_{y=0} (e(x, y) u(x + y)). \quad \square \end{aligned}$$

Remark 2.7. 1. It is important to note that due to the twisting in general $\partial^\gamma \partial^{\gamma'} \neq \partial^{\gamma+\gamma'}$, as can be seen either directly or by the just proved product formula.

2. As in the ordinary pseudodifferential calculus it is in general not true that $P_f^* = P_{f^*}$. However, $\sigma(P_f)^* = \sigma(P_{f^*}) \bmod S^{m-1}(\mathbb{R}^n, \mathcal{A}^\infty)$.

Furthermore, ∂^γ is formally self-adjoint and thus for any *differential* multiplier we have indeed $P_f^* = P_{f^*}$.

2.3.1. *Differential multipliers of order 1 and 2.* Let $e_j, j = 1, \dots, n$ be the canonical basis vectors of \mathbb{R}^n . We abbreviate $\partial_j := \partial^{e_j}$. Then by Prop. 2.6

$$\partial_j u(x) = i^{-1} \partial_{y_j} \big|_{y=0} e^{i\langle Bx, y \rangle} u(x + y) = \left(\frac{1}{i} \partial_{x_j} + b_{jl} x_l \right) u(x), \quad (2.26)$$

where summing over repeated indices is understood. Thus

$$\partial_j \partial_k = -\partial_{x_j} \partial_{x_k} - i b_{js} x_s \partial_{x_k} - i b_{ks} x_s \partial_{x_j} - i b_{kj} + b_{js} b_{kr} x_s x_r, \quad (2.27)$$

in particular

$$[\partial_j, \partial_k] = 2i b_{jk}. \quad (2.28)$$

The product formulas Eq. (2.16), (2.19) now give

$$\begin{aligned} \sigma(\partial_j \cdot \partial_k) &= \sigma(\partial_j) \cdot \sigma(\partial_k) + \frac{1}{i} \sum_{r,s} \partial_{\xi_r}(\xi_j) b_{sr} \partial_{\xi_s}(\xi_k) \\ &= \xi_j \cdot \xi_k + i b_{jk} = \sigma(\partial^{e_j+e_k}) + i b_{jk}. \end{aligned} \quad (2.29)$$

Thus we get the following explicit formula for $\partial^{e_j+e_k}$

$$\partial^{e_j+e_k} = \partial_j \cdot \partial_k - i b_{jk}, \quad (2.30)$$

as well as again the commutator formula $[\partial_j, \partial_k] = 2i b_{jk}$.

2.3.2. *Differential multipliers in dimension $n = 2$.* We consider the case $n = 2$ and the complex Wirtinger derivatives ∂ and $\bar{\partial}$ in this context. Fix $\tau \in \mathbb{C}$ with $\Im \tau > 0$. Put

$$\partial_\tau := \partial_1 + \bar{\tau} \partial_2, \quad \partial_\tau^* = \partial_1 + \tau \partial_2. \quad (2.31)$$

These are constant coefficient differential multipliers and therefore with Eq. (2.28)

$$[\partial_\tau, \partial_\tau^*] = -4 \cdot \Im \tau \cdot b_{12} =: c_\tau. \quad (2.32)$$

We define the Laplacian by

$$\Delta_\tau := \frac{1}{2}(\partial_\tau^* \partial_\tau + \partial_\tau \partial_\tau^*) = \partial_1^2 + |\tau|^2 \partial_2^2 + \Re \tau (\partial_1 \partial_2 + \partial_2 \partial_1). \quad (2.33)$$

Note

$$k^2 \partial_\tau \partial_\tau^* = k^2 \Delta_\tau + \frac{1}{2} k^2 c_\tau, \quad (2.34)$$

$$\partial_\tau k^2 \partial_\tau^* = k^2 \Delta_\tau + (\partial_\tau k^2) \partial_\tau^* + \frac{1}{2} k^2 c_\tau, \quad (2.35)$$

$$\partial_\tau^* k^2 \partial_\tau = k^2 \Delta_\tau + (\partial_\tau^* k^2) \partial_\tau - \frac{1}{2} k^2 c_\tau. \quad (2.36)$$

This symmetric definition of Δ_τ has the advantage that its symbol equals $|\eta|^2 := |\xi_1 + \tau \xi_2|^2$.

3. THE RESOLVENT EXPANSION FOR ELLIPTIC DIFFERENTIAL MULTIPLIERS

3.1. Set-up and formulation of the result. We continue to work in the framework of Section 2. The algebra \mathcal{A} stands for, e.g., either $\mathcal{A}_{\theta'}$, \mathcal{A}_θ , or $\mathcal{A}_{\theta'} \otimes \mathcal{A}_\theta$. ψ stands for the corresponding normalized trace (φ_0 resp. $\varphi_0 \otimes \varphi_0$). We consider the differential multiplier

$$P = P_{\varepsilon_1, \varepsilon_2} := k^2 \Delta_\tau + \varepsilon_1 (\partial_\tau k^2) \partial_\tau^* + \varepsilon_2 (\partial_\tau^* k^2) \partial_\tau + a_0, \quad (3.1)$$

where $a_0 \in \mathcal{A}$ and $\varepsilon_1, \varepsilon_2$ are real parameters.

We want to compute the first three terms in the expansion of the resolvent $(P - \lambda)^{-1}$ in the parameter dependent pseudodifferential calculus. For simplicity we restrict ourselves here to the case $n = 2$ and the special P as above. We emphasize, however, that to a large extent the computation could be done in greater generality; we intend to come back to this in a future publication.

Recall from Subsection 2.3.1 that the symbol of P takes the form

$$\sigma_P(\xi) := a_2(\xi) + a_1(\xi) + a_0, \quad (3.2)$$

where $a_0 \in \mathcal{A}^\infty$ is the same as above and

$$\begin{aligned} a_2(\xi) &= k^2 |\xi_1 + \bar{\tau} \xi_2|^2 =: k^2 |\eta|^2, \\ a_1(\xi) &= \varepsilon_1 (\partial_\tau k^2) \bar{\eta} + \varepsilon_2 (\partial_\tau^* k^2) \eta, & \eta &:= \xi_1 + \bar{\tau} \xi_2, \\ &=: \rho_1 \bar{\eta} + \rho_2 \eta, & \rho_1 &:= \varepsilon_1 \partial_\tau k^2, \rho_2 := \varepsilon_2 \partial_\tau^* k^2. \end{aligned}$$

Below we calculate the resolvent expansion in terms of functions of ξ . The resolvent trace density, at least with respect to the dual trace 2.1.1, is obtained by integration over \mathbb{R}^2 with respect to ξ . Let $f(\eta_1, \eta_2) \in \text{CS}^m(\mathbb{R}^n, \mathcal{A}^\infty)$ be a symbol of order < -2 which is given as a function of $\eta = \xi_1 + \bar{\tau} \xi_2 =: \eta_1 + i \eta_2$. Changing variables we have

$$\int_{\mathbb{R}^2} f(\eta_1, \eta_2) d\xi = \frac{1}{(2\pi)^2 |\Im \tau|} \int_{\mathbb{R}^2} f(\eta_1, \eta_2) d\eta, \quad d\eta := d\eta_1 d\eta_2. \quad (3.3)$$

Furthermore, we note that if $f(\eta_1, \eta_2) = g(|\eta|^2)$ depends only on $|\eta|^2$ then

$$\int_{\mathbb{R}^2} \eta_1^{\alpha_1} \eta_2^{\alpha_2} g(|\eta|^2) d\eta = 0, \quad (3.4)$$

whenever α_1 or α_2 is odd. Furthermore, regardless of the parity of α_j the integral on the left of Eq. (3.4) is invariant under permutations of (α_1, α_2) . In particular

$$\begin{aligned} \int_{\mathbb{R}^2} \eta_k^2 g(|\eta|^2) d\eta &= \frac{1}{2} \int_{\mathbb{R}^2} |\eta|^2 g(|\eta|^2) d\eta, \quad k = 1, 2; \\ \int_{\mathbb{R}^2} \eta^2 f(|\eta|) d\eta &= \int_{\mathbb{R}^2} (\eta_1^2 - \eta_2^2 + 2i\eta_1\eta_2) f(|\eta|) d\eta = 0. \end{aligned} \quad (3.5)$$

For functions f, g of ξ we therefore introduce the notation $f \doteq g$ if f and g have the same ξ -integral, and hence the same η -integral.

Now we are able to state the main result about the resolvent of the multiplier P .

Theorem 3.1. *Let P be the differential multiplier Eq. (3.1) in the pseudodifferential multiplier calculus and denote by $\eta := \xi_1 + \bar{\tau}\xi_2$ the symbol of ∂_τ and let $b(\xi) := (k^2|\eta|^2 - \lambda)^{-1}$.*

Then the resolvent $(P - \lambda)^{-1}$ of P is a parameter dependent pseudodifferential multiplier with polyhomogeneous symbol $b_{-2} + b_{-3} + b_{-4} + \dots$. Up to a function of total ξ -integral 0 we have the following closed formulas for the first three terms in the symbol expansion of $(P - \lambda)^{-1}$:

$$b_{-2} = b = (k^2|\eta|^2 - \lambda)^{-1}, \quad (3.6)$$

$$b_{-3} = -bk^2(\eta\partial_\tau^* + \bar{\eta}\partial_\tau)b - ba_1b, \quad (3.7)$$

$$b_{-4} = (2bk^2|\eta|^2 - 1 - \varepsilon_1 - \varepsilon_2)bk^2\Delta_\tau b \quad (3.8)$$

$$+ \lambda bk^2((\partial_\tau^*b)(\partial_\tau b) + (\partial_\tau b)(\partial_\tau^*b)) \quad (3.9)$$

$$+ \varepsilon_1 \cdot \lambda b(\partial_\tau k^2)b\partial_\tau^*b + \varepsilon_2 \cdot \lambda b(\partial_\tau^*k^2)b\partial_\tau b \quad (3.10)$$

$$+ \varepsilon_1\varepsilon_2 \cdot |\eta|^2b \cdot ((\partial_\tau k^2)b(\partial_\tau^*k^2) + (\partial_\tau^*k^2)b\partial_\tau k^2) \cdot b - ba_0b. \quad (3.11)$$

3.1.1. The second heat coefficient in terms of k^2 . To formulate the main result about the second heat coefficient we introduce the following abbreviations:

$$\begin{aligned} \tilde{\square}^{\Re}(k^2) &= \frac{1}{2}(k^{-2}\partial_\tau k^2 \cdot k^{-2}\partial_\tau^*k^2 + k^{-2}\partial_\tau^*k^2 \cdot k^{-2}\partial_\tau k^2) \\ \tilde{\square}^{\Im}(k^2) &= \frac{1}{2}(k^{-2}\partial_\tau k^2 \cdot k^{-2}\partial_\tau^*k^2 - k^{-2}\partial_\tau^*k^2 \cdot k^{-2}\partial_\tau k^2). \end{aligned} \quad (3.12)$$

Furthermore, denote by $\Delta = k^{-2} \cdot k^2 = e^{\nabla_h}$ the modular operator.

Theorem 3.2. *Let P be the differential multiplier Eq. (3.1). Then there is an asymptotic expansion*

$$\mathrm{Tr}_\psi(ae^{-tP}) \sim_{t \searrow 0} \sum_{j=0}^{\infty} a_{2j}(a, P) t^{j-1}.$$

Here, Tr_ψ is the natural trace on the algebra of pseudodifferential multipliers (cf. Remark 2.4) induced by the (unique) normalized trace ψ on \mathcal{A} , which is invariant under the natural \mathbb{R}^2 -action.

We have

$$a_0(a, P) = \frac{1}{4\pi|\Im\tau|}\psi(ak^{-2}). \quad (3.13)$$

Furthermore, there exist functions $F(u)$, $G^{\Re}(u, v)$, $G^{\Im}(u, v)$ such that the second heat coefficient of P is given by

$$a_2(a, P) = \frac{1}{4\pi|\Im\tau|} \psi \left[a \left(F(\Delta)(k^{-2}\Delta_\tau k^2) - \mathcal{L}_0(\Delta)(k^{-2}a_0) \right. \right. \\ \left. \left. + G^{\Re}(\Delta^{(1)}, \Delta^{(2)})(\tilde{\square}^{\Re}(k^2)) + G^{\Im}(\Delta^{(1)}, \Delta^{(2)})(\tilde{\square}^{\Im}(k^2)) \right) \right], \quad (3.14)$$

The functions F , G^{\Re} , G^{\Im} depend only on P but not on τ . They are linear combinations of simple [¶]divided differences of \log . In particular they are analytic on the Riemann surface of \log . $\mathcal{L}_0(u) = \frac{\log u}{u-1}$ is the generating function of the Bernoulli numbers.

The functions will be given explicitly in Eq. (4.18), (4.20), (4.19), (4.26), (4.30), and (4.27) below.

3.2. Second heat coefficient in terms of $\log k^2$. To express the second heat coefficient in terms of $\log k^2$ we write for $h \in \mathcal{A}^\infty$

$$\begin{aligned} \partial_\tau h \cdot \partial_\tau^* h &=: \square^{\Re}(h) + \square^{\Im}(h) \\ \partial_\tau^* h \cdot \partial_\tau h &=: \square^{\Re}(h) - \square^{\Im}(h). \end{aligned} \quad (3.15)$$

Clearly,

$$\begin{aligned} \square^{\Re}(h) &= \frac{1}{2}(\partial_\tau h \cdot \partial_\tau^* h + \partial_\tau^* h \cdot \partial_\tau h) \\ \square^{\Im}(h) &= \frac{1}{2}(\partial_\tau h \cdot \partial_\tau^* h - \partial_\tau^* h \cdot \partial_\tau h). \end{aligned} \quad (3.16)$$

Corollary 3.3. *There exist entire functions $K(s)$, $H^{\Re}(s, t)$, $H^{\Im}(s, t)$, such that with $h := \log k^2$ the second heat coefficient of P takes the form*

$$a_2(a, P) = \frac{1}{4\pi|\Im\tau|} \psi \left[a \left(K(\nabla)(\Delta_\tau h) - k^{-2}a_0 \right. \right. \\ \left. \left. + H^{\Re}(\nabla^{(1)}, \nabla^{(2)})(\square^{\Re}(h)) + H^{\Im}(\nabla^{(1)}, \nabla^{(2)})(\square^{\Im}(h)) \right) \right], \quad (3.17)$$

The functions will be given explicitly in the proof in Eq. (3.21), (3.22), (3.23).

In the sequel we will always use the following synonyms for variables: $s = \log(u)$, $t = \log(v)$.

Proof. Recall from [FAKH13, Lemma 5.1], see also [CoTr11, Proof of Lemma 3.3], [CoMo14, Sec. 6.1], resp. in the notation of divided differences [LES14, Ex. 3.13]

[¶]That means at most the third divided difference occurs. Recall that the divided differences of a smooth function f are recursively defined by

$$[x_0]f := f(x_0), \quad [x_0, \dots, x_n]f := \frac{1}{x_0 - x_n}([x_0, \dots, x_{n-1}]f - [x_1, \dots, x_n]f).$$

For a review and further references see [LES14, Appendix A].

$$k^{-2} \partial_\tau^{(*)} k^2 = \frac{e^\nabla - 1}{\nabla} \partial_\tau^{(*)} h \quad (3.18)$$

$$k^{-2} \Delta_\tau k^2 = \frac{e^\nabla - 1}{\nabla} \Delta_\tau h + 2[0, \nabla^{(1)}, \nabla^{(1)} + \nabla^{(2)}] \exp(\square^{\mathfrak{R}}(h)), \quad (3.19)$$

where

$$\begin{aligned} [0, s, s+t] \exp &= \frac{(e^{s+t} - 1)s - (e^s - 1)(s+t)}{st(s+t)} \\ &= \frac{(uv - 1) \log(u) - (u - 1) \log(uv)}{\log(u) \log(v) \log(uv)}. \end{aligned} \quad (3.20)$$

I. e., $[\cdot, \cdot, \cdot] \exp$ is the second divided difference of the exponential function.

Inserting these formulas into Eq. (3.14) immediately yields the claim. Moreover, we get the following explicit formulas for the functions $K, H^{\mathfrak{R}}, H^{\mathfrak{J}}$ in terms of $F, G^{\mathfrak{R}}, G^{\mathfrak{J}}$:

$$K(s) = F(u) \frac{u-1}{\log(u)} = F(e^s) \frac{e^s - 1}{s}, \quad (3.21)$$

$$\begin{aligned} H^{\mathfrak{R}}(s, t) &= 2F(uv) \frac{(uv - 1) \log(u) - (u - 1) \log(uv)}{\log(u) \log(v) \log(uv)} \\ &\quad + G^{\mathfrak{R}}(u, v) \frac{u-1}{\log(u)} \frac{v-1}{\log(v)} \\ &= 2F(e^{s+t}) \frac{(e^{s+t} - 1)s - (e^s - 1)(s+t)}{st(s+t)} \\ &\quad + G^{\mathfrak{R}}(e^s, e^t) \frac{e^s - 1}{s} \frac{e^t - 1}{t} \end{aligned} \quad (3.22)$$

$$\begin{aligned} H^{\mathfrak{J}}(s, t) &= G^{\mathfrak{J}}(u, v) \frac{u-1}{\log(u)} \frac{v-1}{\log(v)} \\ &= G^{\mathfrak{J}}(e^s, e^t) \frac{e^s - 1}{s} \frac{e^t - 1}{t}. \end{aligned} \quad (3.23)$$

Recall, that $s = \log(u), t = \log(v)$. \square

3.3. Proof of Theorem 3.1. Let λ be the resolvent parameter and consider the parameter dependent symbol $\sigma_\lambda(\xi) := \sigma_p(\xi) - \lambda =: a'_2 + a_1 + a_0$; $a'_2 := a_2 - \lambda$, which is elliptic in the parameter dependent calculus. To find the resolvent parametrix $B(\lambda)$ we need to solve

$$1 = \sigma_\lambda * \sigma_B \quad (3.24)$$

with respect to the product Eq. (2.16); here we have to treat λ as a variable of order 2.

In the following, unless otherwise said, we understand summation convention over repeated indices running from 1 to 2. The product formula Eq. (2.16) shows that up to terms of order ≤ -1 we have $\sigma_\lambda * \sigma_B = a'_2 \cdot b_{-2}$. This already shows $b := b_{-2} = (a'_2)^{-1}$ Eq. (3.6). Furthermore, note that up to terms of order ≤ -3 the

defect between the product formula Eq. (2.16) and the untwisted product formula (cf. Eq. (2.19)) is given by

$$\frac{1}{i}(\partial_{\xi_j} a_2) b_{jk} (\partial_{\xi_k} b),$$

and this vanishes since b_{jk} is skew and a_2 and b are functions of $|\eta|^2$. Thus up to terms of order ≤ -3 we may employ the usual untwisted product formula and hence we find up to a term of order ≤ -3

$$\sigma_\lambda * \sigma_B = \sigma_\lambda \cdot \sigma_B + \partial_{\xi_j} (a_2 + a_1) \cdot \delta_j \sigma_B + \frac{1}{2} (\partial_{\xi_k} \partial_{\xi_l} a_2) \delta_k \delta_l \sigma_B. \quad (3.25)$$

We rewrite the individual summands on the right as follows:

$$\partial_{\xi_j} a_2 \delta_j = k^2 \delta_{\xi_j} |\eta|^2 \delta_j = k^2 (\bar{\eta} \partial_\tau + \eta \partial_\tau^*), \quad (3.26)$$

$$\partial_{\xi_j} (a_1(\xi)) \delta_j = \partial_{\xi_j} (\rho \bar{\eta} + \rho_2 \eta) \delta_j = \rho \partial_\tau^* + \rho_2 \partial_\tau, \quad (3.27)$$

$$\frac{1}{2} (\partial_{\xi_k} \partial_{\xi_l} a_2) \delta_k \delta_l = \frac{k^2}{2} \delta_{\xi_k} \delta_{\xi_l} (\bar{\eta} \partial_\tau + \eta \partial_\tau^*) = \frac{k^2}{2} (\partial_\tau^* \partial_\tau + \partial_\tau \partial_\tau^*) = k^2 \Delta_\tau. \quad (3.28)$$

Here, the operators $\partial_\tau := \frac{1}{i} \delta_\tau = \frac{1}{i} (\delta_1 + \tau \delta_2)$, $\partial_\tau^* := -\frac{1}{i} \delta_\tau^* = \frac{1}{i} (\delta_1 + \tau \delta_2)$, $\Delta_\tau := \partial_\tau^* \partial_\tau = \partial_\tau \partial_\tau^*$ are the counterparts to the multipliers Eq. (2.31) acting on \mathcal{A} . As opposed to the multipliers where the structure constant Eq. (2.32) may be nonzero the operators ∂_τ and ∂_τ^* commute.

Now we expand $\sigma_B \sim b_{-2} + b_{-3} + b_{-4} + \dots$ into homogeneous terms of order $-2, -3, -4, \dots$. Ordering terms in Eq. (3.25) by homogeneity and writing b for b_{-2} we find

$$\begin{aligned} 1 \sim \sigma_\lambda * \sigma_B &= a'_2 \cdot b \quad (\text{order } 0) \\ &+ a'_2 \cdot b_{-3} + a_1 \cdot b + k^2 (\bar{\eta} \partial_\tau + \eta \partial_\tau^*) b \quad (\text{order } -1) \\ &+ a'_2 \cdot b_{-4} + k^2 (\bar{\eta} \partial_\tau + \eta \partial_\tau^*) b_{-3} + k^2 \Delta_\tau b \quad (\text{order } -2) \\ &+ a_1 \cdot b_{-3} + (\rho_1 \partial_\tau^* + \rho_2 \partial_\tau) b + a_0 \cdot b, \end{aligned} \quad (3.29)$$

and hence

$$b_{-2} = (k^2 |\eta|^2 - \lambda)^{-1}, \quad (3.30)$$

$$b_{-3} = -b k^2 (\bar{\eta} \partial_\tau + \eta \partial_\tau^*) b - b \cdot a_1 \cdot b, \quad (3.31)$$

$$\begin{aligned} b_{-4} &= -b k^2 (\bar{\eta} \partial_\tau + \eta \partial_\tau^*) b_{-3} - b \cdot a_1 \cdot b_{-3} \\ &\quad - b k^2 \Delta_\tau b - b \cdot (\rho_1 \partial_\tau^* + \rho_2 \partial_\tau) b - b \cdot a_0 \cdot b. \end{aligned} \quad (3.32)$$

We rewrite the individual summands of b_{-4} modulo functions of vanishing total integral (cf. the discussion before Theorem 3.1). We make frequent use of the formulas

$$k^2 |\eta|^2 = a'_2 + \lambda = b^{-1} + \lambda, \quad b k^2 |\eta|^2 = 1 + \lambda \cdot b, \quad (3.33)$$

3.3.1. First summand involving b_{-3} . We replace b_{-3} by the rhs of Eq. (3.31) and find for each term

$$\begin{aligned}
& bk^2(\bar{\eta}\partial_\tau + \eta\partial_\tau^*)(bk^2(\bar{\eta}\partial_\tau + \eta\partial_\tau^*)b) \\
& \doteq bk^2\partial_\tau(bk^2|\eta|^2\partial_\tau^*b) + bk^2\partial_\tau^*(bk^2|\eta|^2\partial_\tau b) \\
& = bk^2\partial_\tau((1 + \lambda \cdot b)\partial_\tau^*b) + bk^2\partial_\tau^*((1 + \lambda \cdot b)\partial_\tau b) \\
& = 2bk^2|\eta|^2bk^2\Delta_\tau b + \lambda bk^2((\partial_\tau^*b)\partial_\tau b + (\partial_\tau b)\partial_\tau^*b),
\end{aligned} \tag{3.34}$$

resp.

$$\begin{aligned}
& -bk^2(\bar{\eta}\partial_\tau + \eta\partial_\tau^*)(-b(\rho\bar{\eta} + \rho_2\eta)b) \\
& \doteq k^2|\eta|^2b(\partial_\tau^*(b\rho b) + \partial_\tau(b\rho_2b)) \\
& = bk^2(\partial_\tau^*(\varepsilon_1 b|\eta|^2(\partial_\tau k^2)b)) + bk^2(\partial_\tau(\varepsilon_2 b|\eta|^2(\partial_\tau^* k^2)b)) \\
& = -(\varepsilon_1 + \varepsilon_2)bk^2\Delta_\tau b.
\end{aligned} \tag{3.35}$$

3.3.2. *Second summand* $-ba_1b_{-3} = -b(\rho\bar{\eta} + \rho_2\eta)b_{-3}$ involving b_{-3} . Similarly,

$$\begin{aligned}
& -b(\rho_1\bar{\eta} + \rho_2\eta)(-bk^2(\bar{\eta}\partial_\tau + \eta\partial_\tau^*)b) \\
& \doteq b\rho_1|\eta|^2bk^2\partial_\tau^*b + b\rho_2|\eta|^2bk^2\partial_\tau b \\
& = b\rho(1 + \lambda \cdot b)\partial_\tau^*b + b\rho_2(1 + \lambda \cdot b)\partial_\tau b,
\end{aligned} \tag{3.36}$$

resp.

$$-b(\rho_1\bar{\eta} + \rho_2\eta)(-b(\rho_1\bar{\eta} + \rho_2\eta)b) \doteq |\eta|^2b \cdot (\rho b\rho_2 + \rho_2 b\rho) \cdot b. \tag{3.37}$$

Summing up we have

$$\begin{aligned}
b_{-4} &= (2k^2|\eta|^2b - 1 - \varepsilon_1 - \varepsilon_2)bk^2\Delta_\tau b + \lambda bk^2((\partial_\tau^*b)\partial_\tau b + (\partial_\tau b)\partial_\tau^*b) \\
&\quad - b(\rho_1\partial_\tau^*b + \rho_2\partial_\tau b) + b\rho_1(1 + \lambda b)\partial_\tau^*b + b\rho_2(1 + \lambda b)\partial_\tau b \\
&\quad + |\eta|^2b \cdot (\rho b\rho_2 + \rho_2 b\rho) \cdot b - ba_0b \\
&= (2k^2|\eta|^2b - 1 - \varepsilon_1 - \varepsilon_2)bk^2\Delta_\tau b + \lambda bk^2((\partial_\tau^*b)\partial_\tau b + (\partial_\tau b)\partial_\tau^*b) \\
&\quad + \varepsilon_1\lambda b(\partial_\tau k^2)b\partial_\tau^*b + \varepsilon_2\lambda b(\partial_\tau^* k^2)b\partial_\tau b + |\eta|^2b \cdot (\rho b\rho_2 + \rho_2 b\rho) \cdot b - ba_0b,
\end{aligned}$$

and the Theorem is proved. \square

4. THE INTEGRAL OF b_{-4} AND THE PROOF OF THEOREM 3.2

In the sequel we use decorators $\varepsilon_1, \varepsilon_2$ to emphasize the dependence of an object on the parameter values $\varepsilon_1, \varepsilon_2$. E.g. $F_{\varepsilon_1, \varepsilon_2}$ denotes the universal function F for the multiplier $P_{\varepsilon_1, \varepsilon_2}$, etc.

We will always have to integrate a symbol depending only on $|\eta|^2$ over \mathbb{R}^2 . By change of variables we have for such a function (cf. Eq. (3.3))

$$\int_{\mathbb{R}^2} f(|\eta|^2) d\xi = \frac{1}{4\pi|\mathcal{I}\tau|} \int_0^\infty f(r) dr. \tag{4.1}$$

The existence of the asymptotic expansion in Theorem 3.2 follows from Theorem 3.1 by the usual contour integral argument. Namely,

$$e^{-tP} = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-t\lambda} (\lambda - P)^{-1} d\lambda = -\frac{1}{2\pi i t} \int_{\mathcal{C}} e^{-t\lambda} (\lambda - P)^{-2} d\lambda \tag{4.2}$$

where C is the usual contour encircling the positive real axis. One now argues as in [GL95, Sec. 1.8] to translate the resolvent expansion into the heat expansion. Concretely, one gets for $j > 0$

$$a_{2j}(a, P) = \frac{1}{2\pi i} \int_C e^{-t\lambda} \int_{\mathbb{R}^2} \psi(ab_{-2j}(\eta)) d\xi d\lambda. \quad (4.3)$$

For the constant term we find by Eq. (3.6), and Theorem 5.2

$$\begin{aligned} a_0(a, P) &= -\frac{1}{2\pi i} \int_C e^{-t\lambda} \int_{\mathbb{R}^2} \psi(a(k^2|\eta|^2 - \lambda)^{-2}) d\xi d\lambda \\ &= \frac{-1}{2\pi i \cdot 4\pi|\Im\tau|} \int_C e^{-t\lambda} \int_0^\infty \psi(a(k^2r - \lambda)^{-2}) dr d\lambda \\ &= \frac{1}{4\pi|\Im\tau|} \psi(ak^{-2} \frac{-1}{2\pi i} \int_C e^{-t\lambda} \lambda^{-1} d\lambda) \\ &= \frac{1}{4\pi|\Im\tau|} \psi(ak^{-2}), \end{aligned}$$

cf. Eq. (4.1).

It remains to compute $a_2(a, P)$. In view of Eq. (4.3) this can be done by integrating each summand of b_{-4} and then applying the Rearrangement Lemma [CoMo14, Lemma 6.2], [LES14]. We will adopt a slightly different approach here which will free us from calculating some of the more tedious integrals and which at the same time shows several a priori relations between the functions $F, G^{\Re/\Im}$ for various multipliers. First, we prove the Theorem 3.2 for $P_\gamma = k^2\Delta_\tau \oplus (k^2\Delta_\tau + (\partial_\tau k^2)\partial_\tau^*)$ and the *graded* trace. That is we show that the second heat coefficient of

$$\mathrm{Tr}_{\psi, \gamma}(ae^{-tP_\gamma}) = \mathrm{Tr}_\psi(ae^{-tk^2\Delta_\tau}) - \mathrm{Tr}_\psi(e^{-t(k^2\Delta_\tau + (\partial_\tau k^2)\partial_\tau^*)})$$

is of the form Eq. (4.3) and we need to identify the corresponding functions F, G^{\Re}, G^{\Im} .

Then by employing a simple symmetry argument we will derive a priori relations between the functions $F_{1,1}$ and $F_{0,0}$ resp. $G_{1,1}^{\Re}$ and $G_{0,0}$, Eq. (4.21), (4.22). This will allow us to express the functions for all parameter values of $(\varepsilon_1, \varepsilon_2)$ in terms of those for the graded case.

4.1. Some concrete integrals. For the evaluation of the integrals we use freely the notation and results of [LES14, Sec. 4.5]. Actually, from this paper we will only need a few explicit integrals which can be evaluated using the residue calculus. In order to be self-contained, we collect here the relevant formulas.

Let $p, m \in \mathbb{Z}_+, p \geq 1$, and $\alpha \in \mathbb{Z}_+^p$ a multiindex. Then for $s \in (\mathbb{C} \setminus \mathbb{R}_-)^p$ consider the integral

$$H_\alpha^{(p)}(s, m) := \int_0^\infty x^m (1+x)^{-\alpha_0-1} \prod_{j=1}^p (x+s_j)^{-\alpha_j-1} dx \quad (4.4)$$

$$= \int_0^\infty x^{|\alpha|+p-1-m} (1+x)^{-\alpha_0-1} \prod_{j=1}^p (1+s_j x)^{-\alpha_j-1} dx \quad (4.5)$$

$$= (-1)^{m+|\alpha|+p-1} [1^{\alpha_0+1}, s_1^{\alpha_1+1}, \dots, s_p^{\alpha_p+1}] \mathrm{id}^m \log. \quad (4.6)$$

Here, $[1^{\alpha_0+1}, s_1^{\alpha_1+1}, \dots, s_p^{\alpha_p+1}] \text{id}^m \log$ stands for the divided difference of order $|\alpha| + p + 1$ of the function $x \mapsto x^m \log x$, where the 1 is repeated $\alpha_0 + 1$ times, s_1 is repeated α_1 times etc. We will only need the cases $p = 1, m = 0$ and $p = 2, m \in \{0, 1\}$. Furthermore, we will always have $\alpha_p = 0$.

4.1.1. $p = 0, m = 0, \alpha_0 = \alpha, \alpha_1 = 0$. Then, cf. [LES14, Sec. 5],

$$\begin{aligned} H_{(\alpha,0)}^{(1)}(u, m = 0) &= (-1)^\alpha [1^{\alpha+1}, u] \log \\ &= \frac{(-1)^\alpha}{(u-1)^{\alpha+1}} \left(\log u - \sum_{j=1}^m \frac{(-1)^{j-1}}{j} (u-1)^j \right) =: \mathcal{L}_\alpha(u), \end{aligned} \quad (4.7)$$

which is the modified logarithm introduced in [CoTr11, Sec. 3 and 6].

4.1.2. $p = 2, \alpha_2 = 0$. Using either the calculus of divided differences or by direct verification the function $H_{(\alpha_0, \alpha_1, 0)}^{(2)}(u, v, m = 0)$ can be expressed in terms of the modified logarithm, cf. [LES14, Eq. 5.1]:

$$H_{(\alpha_0, \alpha_1, 0)}^{(2)}(u, v, m = 0) = \frac{(-1)^{\alpha_1+1}}{\alpha_1!} \partial_u^{\alpha_1} \left(\frac{1}{v-u} (\mathcal{L}_{\alpha_0}(v) - \mathcal{L}_{\alpha_0}(u)) \right). \quad (4.8)$$

Finally, for $m = 1$ and $\alpha_1 > 0$ one infers from

$$(\alpha_1 + u \partial_u)(1 + xu)^{-\alpha_1} = \alpha_1(1 + xu)^{-\alpha_1-1}$$

and differentiation under the integral Eq. (4.5) that

$$H_{(\alpha_0, \alpha_1, 0)}^{(2)}(u, v, m = 1) = \frac{1}{\alpha_1} (\alpha_1 + u \partial_u) H_{(\alpha_0, \alpha_1-1, 0)}^{(2)}(u, v, m = 0). \quad (4.9)$$

4.2. **The contributions of the summands of b_{-4} .** Each summand of b_{-4} in Theorem 3.1 is of the form

$$k^2 f_0(|\eta|^2 k^2) \cdot a \cdot f_1(|\eta|^2 k^2)$$

or of the form

$$k^2 f_0(|\eta|^2 k^2) \cdot a \cdot f_1(|\eta|^2 k^2) \cdot b \cdot f_2(|\eta|^2 k^2).$$

In the first case integration over \mathbb{R}^2 with respect to $d\xi$ and application of the Rearrangement Lemma ([CoMo14, Lemma 6.2], [LES14]) yields (up to the factor in front)

$$\int_0^\infty k^2 f_0(x k^2) \cdot a \cdot f_1(x k^2) dx = F(\Delta)(a)$$

with $F(u) = \int_0^\infty f_0(x) f_1(xu) dx$.

In the second case we have similarly

$$\int_0^\infty k^2 f_0(x k^2) \cdot a \cdot f_1(x k^2) \cdot b \cdot f_2(x k^2) dx = G(\Delta^{(1)}, \Delta^{(2)})(a \cdot b)$$

with

$$G(u, v) = \int_0^\infty f_0(x) f_1(xu) f_2(xuv) dx.$$

Often we will have $\Delta(a)$ instead of a . Note that $G(\Delta^{(1)}, \Delta^{(2)})(\Delta(a) \cdot b) = \tilde{G}(\Delta^{(1)}, \Delta^{(2)})(a \cdot b)$ with $\tilde{G}(u, v) = uG(u, v)$.

4.3. The summands Eq. (3.11) of b_{-4} . We first discuss the terms in the last line Eq. (3.11) of the expression for b_{-4} in Theorem 3.1. As usual (cf. [CoMo14, Sec. 6]) we put $\lambda = -1$. Furthermore, by slight abuse of notation we also write $b(x) = (1+x)^{-1}$ under the integral. This is justified since each term $(k^2|\eta|^2 - \lambda)^{-1}$ in b_{-4} contributes, for $\lambda = -1$, to a factor $b(x) = (1+x)^{-1}$ under the integral on the right of Eq. (4.1). In all integrals in the sequel we will tacitly omit the overall factor $\frac{1}{4\pi|\mathfrak{I}\tau|}$.

4.3.1. The summand $-ba_0b$. The total integral of $-ba_0b$ equals, cf. Eq. (4.7),

$$-\int_0^\infty b(x)a_0b(x)dx = -\mathcal{L}_0(\Delta)(k^{-2}a_0). \quad (4.10)$$

4.3.2. The summand $|\eta|^2b \cdot ((\partial_\tau k^2)b(\partial_\tau^* k^2) + (\partial_\tau^* k^2)b\partial_\tau k^2) \cdot b$. We move all k -powers to the left and rewrite this as

$$\dots = |\eta|^2 k^4 b \cdot (\Delta(k^{-2}\partial_\tau k^2)b(k^{-2}\partial_\tau^* k^2) + \Delta(k^{-2}\partial_\tau^* k^2)b(k^{-2}\partial_\tau k^2)) \cdot b.$$

The total integral therefore equals

$$2\Delta^{(1)}g_3(\Delta^{(1)}, \Delta^{(1)}\Delta^{(2)})(\tilde{\square}^{\mathfrak{R}}(k^2)) =: G_3(\Delta^{(1)}, \Delta^{(2)})(\tilde{\square}^{\mathfrak{R}}(k^2))$$

with

$$\begin{aligned} g_3(u, v) &= 2 \int_0^\infty xb(x)b(xu)b(xv)dx \\ &= 2H_{(0,0,0)}^{(2)}(u, v) = -\frac{2}{v-u}(\mathcal{L}_0(v) - \mathcal{L}_0(u)), \end{aligned} \quad (4.11)$$

(cf. Eq. (4.8)) resp.

$$G_3(u, v) = u g_3(u, uv) = 2 \frac{(uv-1)\log(u) - (u-1)\log(uv)}{(u-1)(v-1)(uv-1)}. \quad (4.12)$$

4.4. Proof in the graded case. For the multiplier P_γ ($\varepsilon_1 = 1, \varepsilon_2 = 0$) and the graded trace several summands of b_{-4} vanish and it remains

$$bk^2\Delta_\tau b + b\partial_\tau k^2 b \partial_\tau^* b.$$

4.4.1. $bk^2\Delta_\tau b$. For the first summand we find

$$\begin{aligned} bk^2\Delta_\tau b &= bk^2\partial_\tau\partial_\tau^* b = -|\eta|^2 k^4 b^2 k^{-2}(\Delta_\tau k^2)b \\ &\quad + |\eta|^4 k^6 b^2 \Delta(k^{-2}\partial_\tau k^2)b(k^{-2}\partial_\tau^* k^2)b \\ &\quad + |\eta|^4 k^6 b^2 \Delta(k^{-2}\partial_\tau^* k^2)b(k^{-2}\partial_\tau k^2)b. \end{aligned} \quad (4.13)$$

The total integral of $-|\eta|^2 k^4 b^2 k^{-2}(\Delta_\tau k^2)b$ equals $F_\gamma(\Delta)(k^{-2}\Delta_\tau k^2)$, where, cf. Eq. (4.8),

$$F_\gamma(u) = -\int_0^\infty xb(x)^2b(xu)dx = -\mathcal{L}_1(u) = \frac{\log(u) - u + 1}{(u-1)^2}.$$

The remaining two summands of $\Delta_\tau b$ contribute a summand

$$\Delta^{(1)}g_1(\Delta^{(1)}, \Delta^{(1)}\Delta^{(2)})(\tilde{\square}^{\mathfrak{R}}(k^2)) =: G_1(\Delta^{(1)}, \Delta^{(2)})(\tilde{\square}^{\mathfrak{R}}(k^2))$$

to G^{\Re} , where

$$g_1(u, v) = 2 \int_0^\infty x^2 b(x)^2 b(xu) b(xv) dx \quad (4.14)$$

$$= 2H_{(1,0,0)}^{(2)}(u, v) = \frac{-2}{v-u}(\mathcal{L}_1(v) - \mathcal{L}_1(u)), \quad (4.15)$$

(cf. Eq. (4.8)) resp.

$$\begin{aligned} G_1(u, v) &= u g_1(u, uv) = \frac{2}{v-1}(F_\gamma(uv) - F_\gamma(u)) \\ &= \frac{-2u(uv+u-2)\log(u)}{(u-1)^2(uv-1)^2} + \frac{2\log(v)}{(v-1)(uv-1)^2} + \frac{2u}{(u-1)(uv-1)}. \end{aligned} \quad (4.16)$$

4.4.2. $b\partial_\tau k^2 b\partial_\tau^* b$. Writing $\partial_\tau^* b = -|\eta|^2 b(\partial_\tau k^2) b$ the total integral of this summand equals $\Delta^{(1)} g_2(\Delta^{(1)}, \Delta^{(1)} \Delta^{(2)})(\tilde{\square}^{\Re}(k^2) + \tilde{\square}^{\Im}(k^2)) =: G_2(\Delta^{(1)}, \Delta^{(2)})(\tilde{\square}^{\Re}(k^2) + \tilde{\square}^{\Im}(k^2))$. Since this is the only summand which is not symmetric in $\partial_\tau k^2$ and $\partial_\tau^* k^2$ it is the only one contributing to G_γ^{\Im} . Thus we have $G_\gamma^{\Im}(u, v) = G_2(u, v) = u g_2(u, uv)$ and this function contributes another summand to G_γ^{\Re} . Explicitly,

$$\begin{aligned} g_2(u, v) &= - \int_0^\infty x b(x) b(xu)^2 b(xv) dx \\ &= -H_{(0,1,0)}^{(2)}(u, v, m=1) \\ &= -(1+u\partial_u)H_{(0,0,0)}^{(2)}(u, v, m=0) = (1+u\partial_u)\frac{1}{v-u}(\mathcal{L}_0(v) - \mathcal{L}_0(u)). \end{aligned}$$

In the last line we have used Eq. (4.9). Thus

$$\begin{aligned} G_2(u, v) &= G_\gamma^{\Im}(u, v) = u g_2(u, uv) \\ &= \frac{v\log(v)}{(v-1)^2(uv-1)} + \frac{u\log(u)}{(u-1)^2(uv-1)} - \frac{1}{(u-1)(v-1)}. \end{aligned} \quad (4.17)$$

The proof for P_γ and the graded trace is thus complete. Summing up we have:

$$F_\gamma(u) = \frac{\log(u) - u + 1}{(u-1)^2} \quad (4.18)$$

$$G_\gamma^{\Im}(u, v) = \frac{u\log(u)}{(u-1)^2(uv-1)} + \frac{v\log(v)}{(v-1)^2(uv-1)} - \frac{1}{(u-1)(v-1)} \quad (4.19)$$

$$\begin{aligned} G_\gamma^{\Re}(u, v) &= G_1(u, v) + G_\gamma^{\Im}(u, v) \\ &= \frac{2}{v-1}(F_\gamma(uv) - F_\gamma(u)) + G_\gamma^{\Im}(u, v) \\ &= -\frac{u(uv+2u-3)\log(u)}{(u-1)^2(uv-1)^2} + \frac{(uv^2+v-2)\log(v)}{(v-1)^2(uv-1)^2} \\ &\quad + \frac{uv-2u+1}{(u-1)(v-1)(uv-1)}. \end{aligned} \quad (4.20)$$

4.5. Proof in general by reduction to the graded case. Integrating each summand of b_{-4} over \mathbb{R}^2 and applying the Rearrangement Lemma ([CoMo14, Lemma 6.2], [LES14]) the existence of the functions F, G^{\Re}, G^{\Im} is clear in general. It remains to find an explicit expression for them which will then also prove the remaining properties.

4.5.1. Conjugation argument. Note that Leibniz' rule implies

$$\Delta(P_{0,0}) = k^{-2}k^2\Delta_{\tau}k^2 = P_{1,1} + \Delta_{\tau}(k^2).$$

Hence (cf. Remark 2.4)

$$\mathrm{Tr}_{\psi}(ae^{-tP_{1,1}}) = \mathrm{Tr}_{\psi}((k^2ak^{-2})e^{-t(P_{0,0}-k^2(\Delta_{\tau}k^2)k^{-2})})$$

and comparing with Eq. (3.14) for $P_{1,1}$ and $P_{0,0}$ yields the a priori relations

$$F_{1,1}(u) = uF_{0,0}(u) + \frac{\log(u)}{u-1}, \quad (4.21)$$

$$G_{1,1}^{\Re}(u, v) = uv \cdot G_{0,0}^{\Re}(u, v), \quad (4.22)$$

resp. comparing with Eq. (3.17) yields the a priori relations

$$K_{1,1}(s) = e^s F_{0,0}(s) + 1 \quad (4.23)$$

$$\begin{aligned} H_{1,1}^{\Re}(s, t) &= e^{s+t} \cdot H_{0,0}^{\Re}(s, t) + 2 \frac{(uv-1)\log(u) - (u-1)\log(uv)}{(uv-1)\log(u)\log(v)} \\ &= e^{s+t} \cdot H_{0,0}^{\Re}(s, t) + 2 \frac{(e^{s+t}-1)s - (e^s-1)(s+t)}{(e^{s+t}-1)st}. \end{aligned} \quad (4.24)$$

4.5.2. The one variable function F . The first summand Eq. (3.8) of b_{-4} is the only summand which contributes to the one variable function F . Therefore,

$$F_{\varepsilon_1, \varepsilon_2} = F_{0,0} - (\varepsilon_1 + \varepsilon_2)F_{\gamma}. \quad (4.25)$$

Applying this and Eq. (4.21) to $F_{1,1}$ we find

$$uF_{0,0}(u) + \frac{\log(u)}{u-1} = F_{1,1} = F_{0,0} - 2F_{\gamma},$$

and solving for $F_{0,0}$ we obtain

$$F_{\varepsilon_1, \varepsilon_2}(u) = -\frac{2 + (\varepsilon_1 + \varepsilon_2)(u-1)}{u-1}F_{\gamma}(u) - \frac{\log(u)}{(u-1)^2}. \quad (4.26)$$

4.5.3. The two variable functions. The terms in Eq. (3.10) are the only ones which contribute to $G_{\varepsilon_1, \varepsilon_2}^{\Im}$, thus

$$G_{\varepsilon_1, \varepsilon_2}^{\Im}(u, v) = G_{\varepsilon_1 - \varepsilon_2, 0}^{\Im} = (\varepsilon_2 - \varepsilon_1) \cdot G_{\gamma}^{\Im}. \quad (4.27)$$

Similarly for G^{\Re}

$$G_{\varepsilon_1, \varepsilon_2}^{\Re} = G_{\varepsilon_1 + \varepsilon_2, 0}^{\Re} + \varepsilon_1 \varepsilon_2 \cdot G_3, \quad (4.28)$$

$$G_{\varepsilon, 0}^{\Re} = G_{0,0}^{\Re} - \varepsilon G_{\gamma}^{\Re}. \quad (4.29)$$

As for F this allows to compute $G_{0,0}^{\Re}$ using Eq. (4.22):

$$uvG_{0,0}^{\Re}(u, v) = G_{2,0}^{\Re}(u, v) + G_3(u, v) = G_{0,0}(u, v) - 2G_{\gamma}^{\Re}(u, v) + G_3(u, v),$$

and hence

$$G_{\varepsilon_1, \varepsilon_2}^{\mathfrak{R}}(u, v) = -\frac{2 + (\varepsilon_1 + \varepsilon_2)(uv - 1)}{(uv - 1)} G_{\gamma}^{\mathfrak{R}}(u, v) + \frac{1 + \varepsilon_1 \varepsilon_2 (uv - 1)}{uv - 1} G_3(u, v). \quad (4.30)$$

4.6. Explicit formulas in terms of $\log k^2$. In view of Eq. (3.21), (3.22), and (3.23) the formulas Eq. (4.26), (4.30), and (4.27) can immediately be translated into formulas for the functions $K, H^{\mathfrak{R}}, H^{\mathfrak{J}}$.

$$K_{\gamma}(u) = \frac{u-1}{\log(u)} F_{\gamma}(u) = \frac{\log(u) - u + 1}{\log(u)(u-1)} \quad (4.31)$$

$$= \frac{1}{e^s - 1} - \frac{1}{s} = \frac{1}{2} \coth(s/2) - \frac{1}{2s} - \frac{1}{2}.$$

$$\begin{aligned} K_{\varepsilon_1, \varepsilon_2}(s) &= -\frac{2 + (\varepsilon_1 + \varepsilon_2)(u-1)}{u-1} K_{\gamma}(s) - \frac{1}{u-1} \\ &= -\frac{2 + (\varepsilon_1 + \varepsilon_2 + 1)(u-1)}{(u-1)^2} + \frac{2 + (\varepsilon_1 + \varepsilon_2)(u-1)}{s(u-1)} \end{aligned} \quad (4.32)$$

$$H_{\gamma}^{\mathfrak{J}}(s, t) = \frac{v(u-1)}{s(v-1)(uv-1)} + \frac{u(v-1)}{t(u-1)(uv-1)} - \frac{1}{st} \quad (4.33)$$

$$H_{\varepsilon_1, \varepsilon_2}^{\mathfrak{J}}(s, t) = (\varepsilon_2 - \varepsilon_1) H_{\gamma}^{\mathfrak{J}}(s, t) \quad (4.34)$$

$$H_{\gamma}^{\mathfrak{R}}(s, t) = H_{\gamma}^{\mathfrak{J}}(s, t) \quad (4.35)$$

$$\begin{aligned} & -\frac{2u(v-1)}{(uv-1)(u-1)t} + \frac{2}{s(s+t)} \\ &= \frac{t(u-1)v}{s(s+t)(v-1)(uv-1)} - \frac{su(v-1)}{t(s+t)(u-1)(uv-1)} \\ &+ \frac{u-v}{(s+t)(u-1)(v-1)} + \frac{t-s}{st(s+t)} \\ H_{\varepsilon_1, \varepsilon_2}^{\mathfrak{R}}(s, t) &= -\frac{2 + (\varepsilon_1 + \varepsilon_2)(uv-1)}{uv-1} H_{\gamma}^{\mathfrak{R}}(s, t) \\ &+ 2\varepsilon_1 \varepsilon_2 \cdot \frac{(uv-1)s - (u-1)(s+t)}{s \cdot t \cdot (uv-1)} \end{aligned} \quad (4.36)$$

Note in particular, that

$$H_{1,0}^{\mathfrak{R}}(s, t) = \frac{1+uv}{2} \cdot H_{0,0}(s, t). \quad (4.37)$$

4.7. Comparison with [CoMo14].

4.7.1. Degree 0. When comparing with [CoMo14] one has to take into account that in degree 0 they work with $k\Delta k = \Delta^{1/2}(k^2\Delta)$ instead of $k^2\Delta$; recall that $\Delta = k^{-2} \cdot k^2$ denotes the modular operator and Δ the Laplacian. This means that their one variable functions must be multiplied by $u^{-1/2}$ and their two variable functions must be multiplied by $(uv)^{-1/2}$ before comparing with our functions. Also note that there are slightly different sign and normalization conventions which lead to an overall factor -2 for one variable functions resp. -4 for two variable functions.

The adjusted functions of [CoMo14] are

$$K_0^{\text{CM}}(s) = \frac{2(2 + e^s(s-2) + s)}{s(e^s - 1)^2} = 2 \frac{(u+1) \log(u) - 2u + 2}{\log(u)(u-1)^2} \quad (4.38)$$

$$= -2K_{0,0}(s)$$

$$H_0(s, t)^{\text{CM}} = e^{-(s+t)/2}. \quad (4.39)$$

$$\begin{aligned} & \frac{t(s+t) \cosh(s) - s(s+t) \cosh(t)}{+(s-t)(s+t + \sinh(s) + \sinh(t) - \sinh(s+t))} \\ & \frac{st(s+t) \sinh(s/2) \sinh(t/2) \sinh^2((s+t)/2)}{t^2(u-1)^2 v - s^2 u(v-1)^2 + st(u-v)(uv-1)} \\ & = 8 \cdot \frac{+(t-s)(u-1)(v-1)(uv-1)}{st(s+t)(u-1)(v-1)(uv-1)^2} \\ & = \frac{8}{uv-1} H_{\gamma}^{\mathfrak{R}}(s, t) = -4H_{0,0}^{\mathfrak{R}}(s, t). \end{aligned}$$

4.7.2. *Degree 1.* From [CoMo14, Sec. 3.1.1] we deduce $2K_1^{\text{CM}} = K_0^{\text{CM}} - K_{\gamma}^{\text{CM}}$ (here K_0^{CM} is the unadjusted function from loc. cit.), hence

$$K_1^{\text{CM}}(s) = \frac{4su - 2u^2 + 2}{s(u-1)^2} = -2K_{1,0}(s). \quad (4.40)$$

$$H_1^{\text{CM}}(s, t) = \cosh\left(\frac{s+t}{2}\right) \cdot H_0^{\text{CM}}(s, t) = -4 \frac{uv+1}{2} H_{0,0}^{\mathfrak{R}}(s, t) = -4H_{1,0}^{\mathfrak{R}}(s, t), \quad (4.41)$$

cf. [CoMo14, Eq. 3.3] and Eq. (4.37).

$$\begin{aligned} S^{\text{CM}}(s, t) &= \frac{s+t - t \cosh(s) - s \cosh(t) - \sinh(s) - \sinh(t) - \sinh(s+t)}{st \sinh(\frac{s}{2}) \sinh(\frac{t}{2}) \sinh(\frac{s+t}{2})} \\ &= 4 \left[\frac{1}{st} - \frac{(u-1)v}{s(v-1)(uv-1)} - \frac{u(v-1)}{t(u-1)(uv-1)} \right] \\ &= -4H_{\gamma}^{\mathfrak{J}}(s, t) = 4H_{1,0}^{\mathfrak{J}}(s, t). \end{aligned} \quad (4.42)$$

5. EFFECTIVE PSEUDODIFFERENTIAL OPERATORS AND RESOLVENT

In this section we will use the notation of the Appendix A, in particular A.4. The algebra \mathcal{B} introduced in A.4 may be thought of being A_{θ} or $A_{\theta'}$.

The effective implementation of the pseudodifferential calculus amounts to passing from its realization on multipliers to a direct action on Heisenberg modules (or on the Hilbert space $L^2(\mathcal{B}, \varphi^{\mathcal{B}})$ itself). More concretely, let $\pi : G \rightarrow \mathcal{L}(\mathcal{H})$ be a projective unitary representation of $G = \mathbb{R}^n \times (\mathbb{R}^n)^{\wedge}$. ¶ For a symbol $f \in S^m(\mathbb{R}^n, \mathcal{B}^{\infty})$ the assignment

$$\text{Op}(f) := \int_G f^{\vee}(y) \pi(y) dy$$

provides a homomorphism of the algebra $L_{\sigma}^m(G, \mathcal{B}^{\infty})$ into the (unbounded) operators in the Hilbert space \mathcal{H} , it is clearly a $*$ -representation of the algebra of

¶For definiteness we discuss the pseudodifferential calculus on \mathbb{R}^n only. The extension for groups of the form $\mathbb{R}^n \times F$, with F finite, does not lead to any new aspects.

pseudodifferential multipliers of order ≤ 0 on \mathcal{H} . We will be interested in two cases:

- (1) $\pi = \tilde{\pi} = \pi_0$ resp. $\pi = \pi_w$, where $\tilde{\pi}$ is the representation defined at the end of example A.6 and π_0, π_w are the representations defined in Eq. (1.17), (1.18).
- (2) $\pi = \tilde{\alpha}$ is the unitary representation of \mathbb{R}^2 on the Hilbert space $L^2(\mathcal{B}, \varphi^{\mathcal{B}})$ induced by the normalized action $\tilde{\alpha}$ on \mathcal{B} , cf. Sec. A.7.

During this section, unless otherwise said, π denotes one of these two representations. The operator convention Op refers to π as explained above.

It is well-known that the singular support of the Fourier transform of a symbol is contained in $\{0\}$. This extends easily to $\varphi^{\mathcal{B}}$ -valued symbols.

Lemma 5.1. 1. *Let $f \in S^m(\mathbb{R}^n, \mathcal{B}^\infty)$ be a symbol of order m and let $\chi \in C^\infty(\mathbb{R}^n)$ be a smooth function which vanishes for $|x| \leq \delta_1$ and which is constant 1 for $|x| \geq \delta_2$. Then $\chi \hat{f} \in S(\mathbb{R}^n, \mathcal{B}^\infty)$.*

2. *If $f \in S^m(\mathbb{R}^n \times \Gamma, \mathcal{B}^\infty)$ is a parameter dependent symbol then $\chi \hat{f} \in S(\mathbb{R}^n \times \Gamma, \mathcal{B}^\infty)$.*

For the parameter dependent symbols see the remarks at the beginning of Sec. 2.2.

Proof. The proof is standard. We just sketch the main step in the more general case 2. Let α, β be multiindices, $\gamma \in \mathbb{Z}_+$, and let p be a seminorm on \mathcal{B}^∞ . We may for convenience assume that the parameter λ in the parameter dependent calculus is treated as a covariable of order 1. Otherwise replace λ by $\lambda^{1/\text{ord}(\lambda)}$. Then

$$\begin{aligned} p\left(\xi^\alpha \partial_\xi^\beta \partial_\lambda^\gamma \hat{f}(\xi, \lambda)\right) &= p\left(\int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} \partial_x^\alpha (x^\beta \partial_\lambda^\gamma f(x, \lambda)) \, dx\right) \\ &\leq C \int_{\mathbb{R}^n} (1 + |x| + |\lambda|)^{m - |\gamma| - |\alpha| + |\beta|} \, dx \\ &\leq C(1 + |\lambda|)^{n + m - |\gamma| - |\alpha| + |\beta|}, \end{aligned}$$

as long as $n + m - |\gamma| - |\alpha| + |\beta| < 0$. Since for given γ, β we may choose α as large as we please the claim follows. \square

The next Theorem relates the Hilbert space trace of $\text{Op}(f(\cdot, \lambda))$ to the natural trace on the pseudodifferential multipliers (cf. Remark 2.4).

Theorem 5.2. *Let $n = 2$ and let $f \in S^m(\mathbb{R}^2 \times \Gamma, \mathcal{B}^\infty)$ be a parameter dependent symbol of order $m < -n$. Then $\text{Op}(f)$ is of trace class. Moreover, there is a constant $N(\pi)$ depending only on the representation π such that*

$$\begin{aligned} \text{Tr}\left(\text{Op}(f(\cdot, \lambda))\right) &= N(\pi) \varphi^{\mathcal{B}}(f^\vee(0, \lambda)) + O(\lambda^{-\infty}), \quad \lambda \rightarrow \infty, \\ &= N(\pi) \int_{\mathbb{R}^2} \varphi^{\mathcal{B}}(f(\xi, \lambda)) \, d\xi + O(\lambda^{-\infty}), \quad \lambda \rightarrow \infty. \end{aligned}$$

We have explicitly,

$$N(\pi) = \begin{cases} |\text{rk } \mathcal{E}(g, \theta)| = |c\theta + d|, & \pi = \pi_w, \\ |\text{rk } \mathcal{E}(g, \theta)|^2 = |c\theta + d|^2, & \pi = \tilde{\alpha}. \end{cases}$$

Proof. We preface the proof with two comments.

- (1) First, the precise threshold $m < -n$ for the trace class property is proved as in the case of classical symbols (see e.g., [SHU01, Sec. 27]), and will not be reproduced below.
- (2) Secondly, the rank factor in $N(\tilde{\alpha})$ is an artifact of the fact that the action $\tilde{\alpha}$ is normalized such that its infinitesimal generators are the derivations δ' , cf. Eq. (1.13).

Given now $f \in S^m(\mathbb{R}^2 \times \Gamma, \mathcal{B}^\infty)$ we extend it to $\mathbb{R} \times \mathbb{Z}_c \times \mathbb{R} \times \mathbb{Z}_c$ by putting $g(x_1, \alpha_1, x_2, \alpha_2, \lambda) := f(x_1, x_2, \lambda) \delta_{\alpha_1, 0} \delta_{\alpha_2, 0}$. Then, by definition, $\text{Op}(f(\cdot, \lambda)) = \tilde{\pi}(f^\vee(\cdot, \lambda))$, and

$$\begin{aligned} \tilde{\pi}(f^\vee(\cdot, \lambda)) &= \int_{\mathbb{R}^2} f^\vee(x, \lambda) \tilde{\pi}(x) dx \\ &= |c| \cdot \int_G g(x_1, \alpha_1, x_2, \alpha_2, \lambda) \pi(x_1, \alpha_1, \mu x_2, \alpha_2) d\mu_G(x_1, \alpha_1, x_2, \alpha_2) \\ &= \frac{|c|}{|\mu|} \cdot \int_G g(x_1, \alpha_1, \frac{x_2}{\mu}, \alpha_2; \lambda) \pi(x_1, \alpha_1, x_2, \alpha_2) d\mu_G(x_1, \alpha_1, x_2, \alpha_2). \end{aligned}$$

The constant $|c|$ in the numerator appears since the Haar measure μ_G equals $\frac{1}{|c|} \lambda \otimes \# \otimes \lambda \otimes \#$, cf. Example A.6. The trace formula Theorem A.1 now implies

$$\text{Tr}(\text{Op}(f(\cdot, \lambda))) = \sum_{\ell \in L} \varphi^{\mathcal{B}}(g(\ell, \cdot) \pi(\ell)).$$

By Lemma 5.1 the sum $\sum_{\ell \in L \setminus \{0\}}$ is $O(\lambda^{-\infty})$ as $\lambda \rightarrow \infty$ and the remaining summand for $\ell = 0$ equals indeed $|c\theta + d| \cdot \varphi^{\mathcal{B}}(f^\vee(0, \lambda))$.

Recall that $\tilde{\alpha}_x(\pi(\ell_1 \omega_1 + \ell_2 \omega_2)) = e^{2\pi i \frac{\ell_1 x_1 + \ell_2 x_2}{\text{rk } \mathcal{E}(g, \theta)}} \pi(\ell_1 \omega_1 + \ell_2 \omega_2)$. Thus

$$\begin{aligned} \text{Op}(f) \pi(\ell_1 \omega_1 + \ell_2 \omega_2) &= \int_{\mathbb{R}^2} f^\vee(x) e^{2\pi i \frac{\ell_1 x_1 + \ell_2 x_2}{\text{rk } \mathcal{E}(g, \theta)}} \pi(\ell_1 \omega_1 + \ell_2 \omega_2) dx \\ &= f\left(-\frac{2\pi}{\text{rk } \mathcal{E}(g, \theta)} \ell_1, -\frac{2\pi}{\text{rk } \mathcal{E}(g, \theta)} \ell_2\right) \pi(\ell_1 \omega_1 + \ell_2 \omega_2). \end{aligned}$$

Applying the Poisson summation formula we find

$$\begin{aligned} \text{Tr}(\text{Op}(f(\cdot, \lambda))) &= \sum_{\ell \in \mathbb{Z}^2} f\left(\frac{2\pi}{\text{rk } \mathcal{E}(g, \theta)} \ell_1, \frac{2\pi}{\text{rk } \mathcal{E}(g, \theta)} \ell_2\right) = \frac{\text{rk } \mathcal{E}(g, \theta)^2}{4\pi^2} \sum_{\gamma \in \left(\frac{2\pi}{\text{rk}} \mathbb{Z}^2\right)^\wedge} \hat{f}(\gamma, \lambda) \\ &= \text{rk } \mathcal{E}(g, \theta)^2 \int_{\mathbb{R}^2} f(\xi) d\xi + O(\lambda^{-\infty}). \end{aligned}$$

As above, the sum $\sum_{\gamma \neq 0}$ is $O(\lambda^{-\infty})$ and the summand corresponding to $\gamma = 0$ gives the claimed formula. \square

Theorem 5.3. *Let $k^2 = e^h \in \mathcal{B}^\infty$ be self-adjoint and positive definite and let $a_0 \in \mathcal{B}^\infty$. Furthermore, let*

$$P = \text{Op}(k^2 |\eta|^2 + \varepsilon_1 (\partial_\tau k^2) \bar{\eta} + \varepsilon_2 (\partial_\tau^* k^2) \eta + a_0), \quad \eta = \xi_1 + \bar{\tau} \xi_2,$$

be the effective realization of the differential multiplier Eq. (3.1) w.r.t. the representation π . Then for $a \in \mathcal{B}^\infty$ we have an asymptotic expansion

$$\mathrm{Tr}(ae^{-tP}) \sim_{t \searrow 0} \sum_{j=0}^{\infty} a_{2j}(a, P, \pi) t^{j-1},$$

where

$$a_{2j}(a, P, \pi) = N(\pi) \cdot a_{2j}(a, P) = N(\pi) \cdot \varphi^{\mathcal{B}}(aA_{2j}(P))$$

with $A_{2j}(P) \in \mathcal{B}^\infty$. Explicitly,

$$a_0(a, P, \pi) = \frac{N(\pi)}{4\pi|\Im\tau|} \varphi^{\mathcal{B}}(ak^{-2}), \quad (5.1)$$

and $a_2(a, P)$ is given in Eq. (3.14).

Proof. This follows immediately from Theorem 3.2 and 5.2, cf. also the beginning of Section 4. \square

Example 5.4. We check the normalization constants by calculating explicit examples. Let P be the effective realization of the multiplier $\partial_\tau^* \partial_\tau$ with symbol $|\eta|^2 - \frac{1}{2}c_\tau$, cf. Section 2.3.2.

1. Let $\pi = \tilde{\pi} = \pi_0$. Assume $\Im\tau > 0$. Then, cf. Prop. 1.5 and Eq. (1.20), P is a direct sum of $|c|$ copies of the operator D^*D with

$$D = \frac{\partial}{\partial t} + \lambda t, D^* = -\frac{\partial}{\partial t} + \bar{\lambda} t, \quad \lambda = 2\pi i \mu \bar{\tau},$$

acting on $L^2(\mathbb{R})$. We have $c_\tau = [D, D^*] = 2\Re\lambda = 4\pi\mu\Im\tau$.

$\mu > 0$. Then $\mathrm{spec} D^*D = 4\pi\mu\Im\tau \cdot \mathbb{Z}_+$ and

$$\mathrm{Tr}(e^{-tP}) = \frac{|c|}{1 - e^{-4\pi\mu\Im\tau t}} = \frac{|c\theta + d|}{4\pi\Im\tau} t^{-1} + \frac{|c|}{2} + O(t), \text{ as } t \rightarrow 0. \quad (5.2)$$

$\mu < 0$. Then $\mathrm{spec} D^*D = 4\pi|\mu|\Im\tau \cdot \mathbb{Z}_+ \setminus \{0\}$ and

$$\mathrm{Tr}(e^{-tP}) = \frac{|c|}{e^{4\pi|\mu|\Im\tau t} - 1} = \frac{|c\theta + d|}{4\pi\Im\tau} t^{-1} - \frac{|c|}{2} + O(t), \text{ as } t \rightarrow 0. \quad (5.3)$$

2. Let $\pi = \tilde{\alpha}$. Then $\mathrm{spec} P$ consists of

$$\frac{4\pi^2}{\mathrm{rk} \mathcal{E}(g, \theta)^2} (k_1^2 + |\tau|^2 k_2^2 + 2\Re\tau k_1 k_2), \quad k_1, k_2 \in \mathbb{Z}.$$

Hence by the Poisson summation formula (resp. transformation formula for the θ -function) one finds

$$\begin{aligned} \mathrm{Tr}(e^{-tP}) &= \sum_{k \in \mathbb{Z}^2} e^{-4\pi^2 / \mathrm{rk} (k_1^2 + |\tau|^2 k_2^2 + 2\Re\tau k_1 k_2) t} \\ &= \frac{|\mathrm{rk} \mathcal{E}(g, \theta)|^2}{4\pi|\Im\tau|t} + O(t^\infty), \text{ as } t \rightarrow 0. \end{aligned} \quad (5.4)$$

Eq. (5.2), (5.3), (5.4) are consistent with the values of $N(\pi)$ given in Theorem 5.2 and with Eq. (5.1). Furthermore, by Eq. (3.17) and Theorem 5.3 the constant term

in the heat expansion of $\text{Tr}(e^{-tP})$ equals indeed

$$N(\pi)a_2(a, P) = \frac{N(\pi)}{4\pi\Im\tau} \varphi^{\mathcal{B}}(-\frac{1}{2}(-c_\tau)) = \frac{1}{2}N(\pi)\mu = \frac{|c\theta + d|}{2(\theta + \frac{d}{c})} = \frac{|c|}{2} \text{sgn}(\mu),$$

and this is consistent with Eq. (5.2), (5.3).

APPENDIX A. HEISENBERG EQUIVALENCE BIMODULES AND THE TRACE FORMULA

For the convenience of the reader and to fix some notation we briefly summarize here the main facts about Rieffel's general construction [Rie88, Sec. 2.3] of equivalence bimodules and how it specializes to our setting. We slightly modify his Heisenberg cocycle since for \mathbb{R}^n we prefer to have a skew bicharacter instead of the usual one, cf. Example A.6 below.

A.0.1. Notation. In the sequel all groups will be locally compact abelian (lca). In all our examples, groups will be of the form $\mathbb{R}^n \times F$ with a finite abelian group. Therefore, we will confine ourselves to *elementary* locally compact abelian groups, that is groups of the form $\mathbb{R}^n \times T^m \times \mathbb{Z}^k \times F$. For such groups, the Schwartz space $\mathcal{S}(G)$ and the space of smooth functions on G can be defined as usual. There will be no loss of generality if the reader assumes that $G = \mathbb{R}^n \times F$.

Unless otherwise said, the Haar measure on discrete groups will be the counting measure, the Haar measure on compact groups will usually be normalized to 1. For a finite group F (compact and discrete) the Haar measure will always be specified.

For a lca group G we denote by G^\wedge its Pontryagin dual, and by $\langle \cdot, \cdot \rangle : G \times G^\wedge \rightarrow T = \{z \in \mathbb{C} \mid |z| = 1\} = \mathbb{R}/2\pi\mathbb{Z}$ the bicharacter implementing the duality between G and G^\wedge .

A.1. Poisson summation for discrete cocompact subgroups. Let $\Gamma \subset G$ be a discrete cocompact subgroup. $\text{vol}_G(F_\Gamma) = \text{vol}_G(G/\Gamma)$ denotes the G -volume of a measurable fundamental domain for the action of Γ on G . With the Haar measure on the compact quotient normalized to 1 we have for $f \in \mathcal{S}(G)$

$$\int_G f(x) dx = \text{vol}_G(F_\Gamma) \cdot \int_{G/\Gamma} \sum_{\gamma \in \Gamma} f(x + \gamma) d\dot{x}. \quad (\text{A.1})$$

Let $\Gamma^\perp \subset G^\wedge$ be the discrete group consisting of those $y \in G^\wedge$ with $\langle \gamma, y \rangle = 1$ for all $\gamma \in \Gamma$. Clearly, Γ^\perp is naturally isomorphic to $(G/\Gamma)^\wedge$. Then for $f \in \mathcal{S}(G)$ one has the *Poisson summation formula*

$$\sum_{\gamma \in \Gamma} f(\gamma) = \frac{1}{\text{vol}_G(F_\Gamma)} \cdot \sum_{\alpha \in \Gamma^\perp} \hat{f}(\alpha). \quad (\text{A.2})$$

A.2. Covariance algebras twisted by a cocycle. Let (\mathcal{A}, G, α) be a C^* -dynamical system. By \mathcal{A}^∞ we denote as usual the smooth subalgebra of \mathcal{A} w.r.t. the action α . Let $\omega : G \times G \rightarrow T$ be a smooth *cocycle* (aka multiplier [KLE62]). That is, ω is a smooth map satisfying

$$\omega(x, 0_G) = \omega(0_G, x) = 1, \quad (\text{A.3})$$

$$\omega(x, y)\omega(x + y, z) = \omega(y, z)\omega(x, y + z). \quad (\text{A.4})$$

Applying this with $y = -x, z = x$ we see $\omega(x, -x) = \omega(-x, x)$.

Define a product and involution on $\mathcal{S}(G, \mathcal{A}^\infty)$ as follows:

$$f^*(x) = \overline{\omega(x, -x)} \alpha_x(f(-x)^*), \quad (\text{A.5})$$

$$(f * g)(x) = \int_G f(y) \alpha_y(g(x - y)) \omega(y, x - y) dy. \quad (\text{A.6})$$

With this product and involution $\mathcal{S}(G, \mathcal{A}^\infty)$ becomes a $*$ -algebra. A canonical representation of this $*$ -algebra is obtained as follows: $\mathcal{S}(G, \mathcal{A}^\infty)$ is a pre- C^* -module with \mathcal{A}^∞ -valued inner product

$$\langle f, g \rangle = \int_G f(x)^* g(x) dx. \quad (\text{A.7})$$

Put

$$(\alpha f)(t) = \alpha_{-t}(a)f(t), \quad a \in \mathcal{A}^\infty; \quad (\text{A.8})$$

$$(U_y f)(t) = \omega(y, t - y)f(t - y). \quad (\text{A.9})$$

Then $U_y, y \in G$, is a projective family of unitaries which implements the group of automorphisms $\alpha_y, y \in G$:

$$U_x^* = \overline{\omega(x, -x)} U_{-x}, \quad U_x U_y = \omega(x, y) U_{x+y}, \quad x, y \in G, \quad (\text{A.10})$$

$$U_x a U_x^* = \alpha_x(a), \quad a \in \mathcal{A}^\infty. \quad (\text{A.11})$$

By associating to $f \in \mathcal{S}(G, \mathcal{A}^\infty)$ the multiplier $M_f = \int_G f(x) U_x dx$ we obtain the *left regular representation* of the ω -twisted convolution algebra $\mathcal{S}(G, \mathcal{A}^\infty)$ on the Hilbert C^* -module $L^2(G, \mathcal{A})$. The definition of Eq. (A.9) is deliberately chosen such that $U_y \circ M_f = M_{U_y f}$. This differs from the definition in [KLE62].

A.2.1. Dual Trace. If ψ is an α -invariant (finite) trace on \mathcal{A} then the *dual trace* $\widehat{\psi}$ on $\mathcal{S}(G, \mathcal{A}^\infty)$ is given by

$$\widehat{\psi}(f) = \psi(f(0)) = \int_{G^\wedge} \psi(\widehat{f}(\xi)) d\xi. \quad (\text{A.12})$$

In particular, on the ω -twisted convolution algebra $\mathcal{S}(G)$ (that is $\mathcal{A} = \mathbb{C}$ with trivial G -action) we have the trace

$$\widehat{\psi}(f) = f(0) = \int_{G^\wedge} \widehat{f}(\xi) d\xi. \quad (\text{A.13})$$

In fact, the function f can be expressed as

$$\widehat{\psi}(U_x^* f) = (U_x^* f)(0) = f(x). \quad (\text{A.14})$$

A.3. Rieffel's construction of Heisenberg module. Let M be a lca group, $G := M \times M^\wedge, \langle \cdot, \cdot \rangle : G \rightarrow \mathbb{T}$ the duality pairing. We choose the Haar measure on G to be $\mu_G := \mu_M \otimes \mu_{M^\wedge}$ with a Haar measure μ_M on M and its corresponding Plancherel measure μ_{M^\wedge} on M^\wedge . This Haar measure on G is self-dual w.r.t. the bicharacter $\rho(x, y) := \langle x_1, y_2 \rangle \overline{\langle y_1, x_2 \rangle}, x, y \in G$. Furthermore, the Fourier transform which sends $f \in \mathcal{S}(G)$ to $\widehat{f}(\xi) := \int_G \rho(\xi, x) f(x) d\mu_G(x)$ is self-inverse.

We prefer to have a skew symmetric bicharacter on the \mathbb{R}^n factor of G . To allow for some flexibility we thus define the Heisenberg representation of G on $L^2(M)$ as follows. Fix a bicharacter $\lambda : M \times M^\wedge \rightarrow \mathbb{T}$ and put

$$(\pi(\mathbf{y})u)(t) := \lambda(\mathbf{y})\langle t, \mathbf{y}_2 \rangle u(t + \mathbf{y}_1), \quad \mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2) \in G, t \in M. \quad (\text{A.15})$$

One has

$$\pi(\mathbf{x})\pi(\mathbf{y}) = e(\mathbf{x}, \mathbf{y})\pi(\mathbf{x} + \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in G, \quad (\text{A.16})$$

where $e(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}_1, \mathbf{y}_2 \rangle \overline{\lambda(\mathbf{x}_1, \mathbf{y}_2)} \lambda(\mathbf{y}_1, \mathbf{x}_2)$ is a cocycle satisfying

$$e(\mathbf{x}, \mathbf{y}) \overline{e(\mathbf{y}, \mathbf{x})} = \rho(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}_1, \mathbf{y}_2 \rangle \overline{\langle \mathbf{y}_1, \mathbf{x}_2 \rangle}, \quad \mathbf{x}, \mathbf{y} \in G. \quad (\text{A.17})$$

Note that ρ is independent of the choice of λ and equals therefore the bicharacter induced by the canonical Heisenberg cocycle $\beta(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}_1, \mathbf{y}_2 \rangle$. Furthermore, $\pi(\mathbf{x})\pi(\mathbf{y}) = \rho(\mathbf{x}, \mathbf{y})\pi(\mathbf{y})\pi(\mathbf{x})$.

The representation π integrates to a representation of the e -twisted convolution algebra $\mathcal{S}(G)$ on $L^2(M)$,

$$\pi(f) := \int_G f(\mathbf{y})\pi(\mathbf{y})d\mathbf{y}, \quad f \in \mathcal{S}(G). \quad (\text{A.18})$$

A.4. The algebras associated to a lattice. Let $L \subset G$ be a lattice, that is a discrete cocompact subgroup and let $L^\perp \subset G$ be the dual lattice with respect to the skew bicharacter ρ . Let \mathcal{B} be the C^* -algebra generated by $\pi(l), l \in L$ and let \mathcal{A} be the C^* -algebra generated by $\pi(l')^{\text{op}}, l' \in L^\perp$.

$x \mapsto \pi(x) \cdot \pi(x)^*$ defines a G -action α on \mathcal{B} , which is the integrated form of

$$\pi(x)\pi(l)\pi(x)^* = \rho(x, l)\pi(l).$$

\mathcal{B}^∞ consists of the smooth vectors w.r.t. this action. Elements of \mathcal{B}^∞ are of the form $\sum_{l \in L} f(l)\pi(l)$ with $f \in \mathcal{S}(L)$. Thus \mathcal{B}^∞ is nothing but the e -twisted convolution algebra $\mathcal{S}(L)$ with the cocycle e restricted to L . Since L is discrete, the algebra \mathcal{B}^∞ is unital with unit $\delta_0(l) = \delta_{l0}$. The dual trace $\varphi^{\mathcal{B}}(f) = f(0)$ is a normalized trace on \mathcal{B} .

Needless to say, the same remarks apply to \mathcal{A} .

A.4.1. The integrated action of $\mathcal{S}(G, \mathcal{B}^\infty)$ on $L^2(M)$. We now apply the construction of Sec. A.2 to the dynamical system (\mathcal{B}, G, α) . The representation Eq. (A.18) extends to a representation of the e -twisted convolution algebra $\mathcal{S}(G, \mathcal{B}^\infty)$ on the Heisenberg module $L^2(M)$. Namely, for $f \in \mathcal{S}(G, \mathcal{B}^\infty) = \mathcal{S}(G \times L)$, we put, by slight abuse of notation:

$$\begin{aligned} \pi(f) &:= \int_G \pi(f(x, \cdot))\pi(x)dx = \int_G \sum_{l \in L} f(x, l)\pi(l)\pi(x)dx \\ &= \int_G \sum_{l \in L} f(x, l)e(l, x)\pi(l + x)dx = \sum_{l \in L} \int_G f(x - l, l)e(l, x - l)\pi(x)dx \quad (\text{A.19}) \\ &= \int_G \left(\sum_{l \in L} f(x - l, l)e(l, x - l) \right) \pi(x)dx =: \pi(\Phi_L(f)), \end{aligned}$$

with

$$\Phi_L(f)(x) = \sum_{\ell \in L} f(x - \ell, \ell) e(\ell, x - \ell). \quad (\text{A.20})$$

It is not hard to see that Φ_L maps $\mathcal{S}(G \times L)$ continuously to $\mathcal{S}(G)$, hence the action of $\pi(f)$ is in fact given by the action of the *scalar* Schwartz function $\Phi_L(f)$.

Thus we have the following commutative diagram of maps

$$\begin{array}{ccc} \mathcal{S}(G \times L) & & \\ \Phi_L \downarrow & \searrow \pi & \\ \mathcal{S}(G) & \longrightarrow & \mathcal{L}(L^2(M)). \end{array}$$

A.4.2. The trace formula.

Theorem A.1. *Let $G = M \times M^\wedge$ be an elementary group, $L \subset G$ a discrete cocompact subgroup and let π be the Heisenberg representation as outlined above.*

1. *For $f \in \mathcal{S}(G, \mathcal{B}^\infty) \simeq \mathcal{S}(G \times L)$ the action $\pi(f)$ on $L^2(M)$ is an integral operator with Schwartz kernel $k \in \mathcal{S}(M \times M)$. Concretely, with $g = \Phi_L(f) \in \mathcal{S}(G)$ the Schwartz kernel is given by*

$$k(t, s) = (g(s - t, \cdot) \lambda(s - t, \cdot))^\vee(t), \quad t, s \in M. \quad (\text{A.21})$$

Furthermore, we have the trace formula

$$\text{Tr}(\pi(f)) = \widehat{\psi}(\Phi_L(f)) = \sum_{\ell \in L} f(\ell, -\ell) e(-\ell, \ell) = \sum_{\ell \in L} \psi^\mathcal{B}(f(\ell, \cdot) \pi(\ell)). \quad (\text{A.22})$$

2. *For any integral operator K on $L^2(M)$ with kernel $k \in \mathcal{S}(M \times M)$ the function*

$$g(x) = \text{Tr}(\pi(x)^* K) = \overline{\lambda(x_1, x_2)} \int_M k(t, x_1 + t) \overline{\langle t, x_2 \rangle} dt \quad (\text{A.23})$$

is a Schwartz function such that $K = \pi(g)$.

Proof. 1. We have already seen in Eq. (A.19) that $\pi(f) = \pi(\Phi_L(f))$ and that $g = \Phi_L(f)$ is in the Schwartz space $\mathcal{S}(G)$. In terms of g the action of $\pi(f)$ on $L^2(M)$ is indeed given by

$$\begin{aligned} (\pi(f)u)(t) &= \int_G g(x_1, x_2) \lambda(x_1, x_2) \langle t, x_2 \rangle u(t + x_1) dx \\ &= \int_M \left(\int_{M^\wedge} g(s - t, x_2) \lambda(s - t, x_2) \langle t, x_2 \rangle dx_2 \right) u(s) ds, \end{aligned}$$

and so its Schwartz kernel is $k(t, s) = (g(s - t, \cdot) \lambda(s - t, \cdot))^\vee(t)$, $t, s \in M$.

Clearly, $k \in \mathcal{S}(M \times M)$. It is well-known that integral operators whose Schwartz kernels are Schwartz functions are trace class. Applying Mercer's Theorem [BCD⁺72, Prop. 3.1.1 p. 102] for trace class operators, one obtains

$$\begin{aligned} \text{Tr}(\pi(f)) &= \int_M k(t, t) dt = \int_M (g(0, \cdot) \lambda(0, \cdot))^\vee(t) dt \\ &= g(0_G) = \widehat{\psi}(\Phi_L(f)) = \sum_{\ell \in L} f(-\ell, \ell) e(\ell, -\ell). \end{aligned}$$

To see the last equality of Eq. (A.22) one computes

$$\sum_{\tilde{\ell} \in L} f(\ell, \tilde{\ell}) \pi(\tilde{\ell}) \pi(\ell) = \sum_{\tilde{\ell} \in L} f(\ell, \tilde{\ell}) e(\tilde{\ell}, \ell) \pi(\tilde{\ell} + \ell) = \sum_{\tilde{\ell} \in L} f(\ell, \tilde{\ell} - \ell) e(\tilde{\ell} - \ell, \ell) \pi(\tilde{\ell}),$$

thus $\varphi^{\mathcal{B}}(f(\ell, \cdot) \pi(\ell)) = f(-\ell, \ell) e(-\ell, \ell)$ and 1. is proved.

2. Given $k \in \mathcal{S}(M \times M)$ we invert Eq. (A.21) and put

$$g(x_1, x_2) \lambda(x_1, x_2) := k(\cdot, x_1 + \cdot)^\wedge(x_2) = \int_M k(t, x_1 + t) \overline{\langle t, x_2 \rangle} dt; \quad (\text{A.24})$$

certainly $g \in \mathcal{S}(G)$. \square

Remark A.2. The equations Eq. (A.21), (A.23) provide a linear isomorphism and its inverse between the e -twisted convolution algebra $\mathcal{S}(G)$ and the algebra of integral operators $\mathcal{S}(M \times M)$ on $L^2(M)$ with product and involution given by

$$(k_1 * k_2)(x_1, x_2) = \int_M k_1(x_1, y) k_2(y, x_2) dy, \\ k^*(x_1, x_2) = \overline{k(x_2, x_1)}.$$

As an application we reprove the orthogonality relations for the Heisenberg representation.

Corollary A.3 (Orthogonality Relations). *Let $K_1, K_2 \in \mathcal{L}(L^2(M))$ be integral operators with Schwartz kernels $k_1, k_2 \in \mathcal{S}(M \times M)$ of Schwartz class. Then the Fourier transform of the function*

$$F(x) := \overline{\text{Tr}(\pi(x)^* K_1)} \text{Tr}(\pi(x)^* K_2) \quad (\text{A.25})$$

is given by

$$F^\wedge(\xi) = \text{Tr}(K_1^* \pi(\xi) K_2 \pi(\xi)^*). \quad (\text{A.26})$$

In particular

$$\int_G \overline{\text{Tr}(\pi(x)^* K_1)} \text{Tr}(\pi(x)^* K_2) dx = \text{Tr}(K_1^* K_2), \quad (\text{A.27})$$

and for the discrete cocompact subgroup L we have

$$\sum_{\ell \in L} \overline{\text{Tr}(\pi(\ell)^* K_1)} \text{Tr}(\pi(\ell)^* K_2) = \frac{1}{\text{vol}_G(F_L)} \sum_{\xi \in L^\perp} \text{Tr}(K_1^* \pi(\xi) K_2 \pi(\xi)^*). \quad (\text{A.28})$$

In particular, for Schwartz functions $f, g, h, k \in \mathcal{S}(M)$

$$\int_G \overline{(f, \pi(x)^* g)_{L^2}} (h, \pi(x)^* k)_{L^2} dx = \overline{(f, h)_{L^2}} (g, k)_{L^2}. \quad (\text{A.29})$$

Proof. Eq. (A.16) implies $\pi(\xi)^* \pi(y)^* \pi(\xi) = \rho(\xi, y) \pi(y)^*$. Writing $K_1 = \pi(f_1)$, $K_2 = \pi(f_2)$ Theorem A.1 and Eq. (A.14) imply

$$\begin{aligned} F^\wedge(\xi) &= \int_G \rho(\xi, y) F(y) dy = \int_G \overline{\text{Tr}(\pi(y)^* \pi(f_1))} \text{Tr}(\pi(y)^* (\pi(\xi) \pi(f_2) \pi(\xi)^*)) dy \\ &= \int_G \overline{f_1(y)} (U_\xi f_2 U_\xi^*)(y) dy = (f_1^* * (U_\xi f_2 U_\xi^*))(0) \\ &= \widehat{\psi}(f_1^* * (U_\xi f_2 U_\xi^*)) = \text{Tr}(\pi(f_1^* * (U_\xi f_2 U_\xi^*))) = \text{Tr}(K_1^* \pi(\xi) K_2 \pi(\xi)^*), \end{aligned}$$

proving Eq. (A.26). Specializing $\xi = 0$ gives Eq. (A.27) and applying the Poisson summation formula to F gives Eq. (A.28). Finally, Eq. (A.29) is obtained from Eq. (A.27) with $K_1 = (f, \cdot)_{L^2} g$, $K_2 = (h, \cdot)_{L^2} k$. \square

A.5. Inner product. Put for $f, g \in \mathcal{S}(M)$

$$\begin{aligned} {}_{\mathcal{B}}\langle f, g \rangle &= \text{vol}_G(F_L) \cdot \sum_{l \in L} (\pi(l)g, f)_{L^2} \pi(l), \\ \langle f, g \rangle_{\mathcal{A}} &= \sum_{\xi \in L^\perp} (f\pi(\xi)^{\text{op}}, g)_{L^2} \pi(\xi)^{\text{op}} = \sum_{\xi \in L^\perp} (\pi(\xi)f, g)_{L^2} \pi(\xi)^{\text{op}}. \end{aligned}$$

The orthogonality relations Cor. A.3 then imply that for any $f, g, h \in \mathcal{S}(M)$

$$f \cdot \langle g, h \rangle_{\mathcal{A}} = {}_{\mathcal{B}}\langle f, g \rangle \cdot h,$$

see [Rie88, Sec. 2]. Furthermore, with the normalized traces $\varphi^{\mathcal{A}}, \varphi^{\mathcal{B}}$ we have

$$\frac{1}{\text{vol}_G(F_L)} \cdot \varphi^{\mathcal{B}}({}_{\mathcal{B}}\langle g, f \rangle) = (f, g)_{L^2} = \varphi^{\mathcal{A}}(\langle f, g \rangle_{\mathcal{A}}).$$

The constant $\text{vol}_G(F_L)$ comes from the Poisson summation formula. In our main example it is $\frac{1}{|c\theta+d|}$.

A.6. Example. As an important example we specialize what we have presented so far in this Appendix to the situation considered in the main body of the paper: namely let $M = \mathbb{R} \times \mathbb{Z}/c\mathbb{Z}$ with $\mu_M = \nu \otimes \#$, i.e., Lebesgue measure tensored by the counting measure. The pairing

$$\langle x_1, \alpha_1; x_2, \alpha_2 \rangle := e^{2\pi i(x_1 x_2 + \alpha_1 \alpha_2 / c)}, \quad (x_1, \alpha_1), (x_2, \alpha_2) \in M, \quad (\text{A.30})$$

identifies M^\wedge with M . The Plancherel measure is $\mu_{M^\wedge} = \nu \otimes \frac{1}{|c|}\#$. With

$$\lambda(x_1, \alpha_1; x_2, \alpha_2) = \lambda(x_1, x_2) = \langle x_1, x_2/2 \rangle, \quad (\text{A.31})$$

we have

$$e(\mathbf{x}, \mathbf{y}) = e((x_1, \alpha_1), (x_2, \alpha_2); (y_1, \beta_1), (y_2, \beta_2)) = e^{2\pi i\left(\frac{x_1 y_2 - x_2 y_1}{2} + \frac{\alpha_1 \beta_2}{c}\right)}, \quad (\text{A.32})$$

$$\rho(\mathbf{x}, \mathbf{y}) = e(\mathbf{x}, \mathbf{y}) \overline{e(\mathbf{y}, \mathbf{x})} = e^{2\pi i\left(x_1 y_2 - x_2 y_1 + \frac{\alpha_1 \beta_2 - \alpha_2 \beta_1}{c}\right)}, \quad (\text{A.33})$$

and hence the projective representation π of G on $L^2(M)$ is explicitly given by

$$\begin{aligned} (\pi(y_1, \beta_1; y_2, \beta_2)u)(t, \alpha) &= \lambda(y_1, y_2) \cdot \langle (t, \alpha), (y_2, \beta_2) \rangle \cdot u(t + y_1, \alpha + \beta_1) \\ &= e^{2\pi i\left(\frac{y_1 y_2}{2} + t y_2 + \frac{\alpha \beta_2}{c}\right)} \cdot u(t + y_1, \alpha + \beta_1), \end{aligned} \quad (\text{A.34})$$

and infinitesimally (cf. Eq. (1.20))

$$\frac{\partial}{\partial y_1} \Big|_{y=0} (\pi(\mathbf{y})u)(t, \alpha) = \frac{\partial}{\partial t} u(t, \alpha), \quad (\text{A.35})$$

$$\frac{\partial}{\partial y_2} \Big|_{y=0} (\pi(\mathbf{y})u)(t, \alpha) = 2\pi i \cdot t \cdot u(t, \alpha). \quad (\text{A.36})$$

Consider the following discrete cocompact subgroups of $G = M \times M^{(\wedge)}$:

$$L = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2, \quad \omega_1 = (0, 0; \frac{1}{c\theta+d}, -1), \quad \omega_2 = (-\frac{1}{c}, -\alpha; 0, 0), \quad (\text{A.37})$$

$$L^\perp = \mathbb{Z}\tilde{\omega}_1 \oplus \mathbb{Z}\tilde{\omega}_2, \quad \tilde{\omega}_1 = (0, 0; 1, -d), \quad \tilde{\omega}_2 = (-\theta - \frac{d}{c}, -1; 0, 0). \quad (\text{A.38})$$

One checks that L^\perp is indeed the group of those $\xi \in G$ such that $\rho(\xi, \ell) = 1$ for all $\ell \in L$ and vice versa. Furthermore, w.r.t. the self-dual Haar measure $\lambda \otimes \# \otimes \lambda \otimes \# / |c|$ one has

$$\text{vol}_G(F_L) = \frac{1}{|c\theta + d|} = \text{vol}_G(F_{L^\perp})^{-1}. \quad (\text{A.39})$$

Comparing with Eq. (1.7) we see that $V_1 = \pi(\omega_1)$, $V_2 = \pi(\omega_2)$ and furthermore

$$V_2^l V_1^k = e^{2\pi i k l \left(\frac{1}{c(c\theta+d)} + \theta' \right)} \cdot \pi(k\omega_1 + l\omega_2) = e^{2\pi i k l \theta'} \cdot V_1^k V_2^l.$$

Similarly, comparing with Eq. (1.6) we find $U_1 = \pi(\tilde{\omega}_1)^{\text{op}}$, $U_2 = \pi(\tilde{\omega}_2)^{\text{op}}$ and

$$\begin{aligned} U_2^l U_1^k &= \pi(l\tilde{\omega}_2)^{\text{op}} \pi(k\tilde{\omega}_1)^{\text{op}} = (\pi(k\tilde{\omega}_1) \pi(l\tilde{\omega}_2))^{\text{op}} \\ &= e^{2\pi i k l \frac{\theta+d/c}{2}} \cdot \pi(k\tilde{\omega}_1 + l\tilde{\omega}_2)^{\text{op}} = e^{2\pi i k l \theta} (\pi(l\tilde{\omega}_2) \pi(k\tilde{\omega}_1))^{\text{op}} \\ &= e^{2\pi i k l \theta} \cdot U_1^k U_2^l. \end{aligned}$$

The natural action α of G on the algebras $\mathcal{B}^\infty, \mathcal{A}^\infty$ is given by

$$\begin{aligned} \alpha_x(V_1) &= e^{2\pi i \left(\frac{x_1}{c\theta+d} - \frac{\alpha_1}{c} \right)} \cdot V_1, \quad x = (x_1, \alpha_1; x_2, \alpha_2), \\ \alpha_x(V_2) &= e^{2\pi i \frac{x_1 + a\alpha_2}{c}} \cdot V_2, \\ \alpha_x(U_1) &= e^{2\pi i \left(x_1 - \frac{\alpha_1}{c} \right)} \cdot U_1, \\ \alpha_x(U_2) &= e^{2\pi i \left((\theta + \frac{d}{c})x_2 + \frac{\alpha_2}{c} \right)} \cdot U_2. \end{aligned}$$

Putting $\tilde{\pi}(x_1, x_2) := \pi(x_1, 0; \mu x_2, 0)$, $\mu = \frac{1}{\theta + \frac{d}{c}}$ and $\tilde{\alpha}_{x_1, x_2}(b) := \tilde{\pi}(x_1, x_2) \cdot b \cdot \tilde{\pi}(x_1, x_2)^*$ we find, cf. Eq. (1.17), (1.20), indeed

$$\begin{aligned} (\tilde{\pi}(x)u)(t, \alpha) &= e^{\pi i \mu (x_1 x_2 + 2t x_2)} u(t + x_1), \\ \tilde{\alpha}_{x_1, x_2}(V_j) &= e^{\frac{2\pi i}{c\theta+d} x_j} \cdot V_j, \quad j = 1, 2, \\ \tilde{\alpha}_{x_1, x_2}(U_j) &= e^{2\pi i x_j} \cdot U_j, \quad j = 1, 2. \end{aligned} \quad (\text{A.40})$$

A.7. The trace formula for “trivial vector bundles”. For completeness we discuss here the analog of the trace formula Theorem A.1 for the trivial \mathcal{B} -vector bundle.

With the notation of Example A.6 let $\tilde{\alpha}$ be the normalized action of G on \mathcal{B} . The family $(\pi(\ell))_{\ell \in L}$ is an orthonormal basis of the Hilbert space $L^2(\mathcal{B}, \varphi^\mathcal{B})$ which is the completion of \mathcal{B} with respect to the scalar product $(a, b)_{L^2} = \varphi^\mathcal{B}(a^* b)$. Furthermore, the action $\tilde{\alpha}$ is at the same time a unitary representation of G on $L^2(\mathcal{B}, \varphi^\mathcal{B})$. Letting \mathcal{B} act on the left on $L^2(\mathcal{B}, \varphi^\mathcal{B})$ the unitary action $\tilde{\alpha}$ on $L^2(\mathcal{B}, \varphi^\mathcal{B})$ implements the action $\tilde{\alpha}$ on \mathcal{B} . Clearly, $\tilde{\alpha}$ integrates in the usual way to a representation of the (untwisted) convolution algebra $\mathcal{S}(G, \mathcal{B}^\infty) \simeq \mathcal{S}(G \times L)$. A calculation similar to Eq. (A.19) gives

$$\tilde{\alpha}(f)(\pi(l)) = \int_G f(x) \tilde{\alpha}_x(\pi(l)) dx = \int_G f(x) \rho(x, l) \pi(l) dx$$

$$\begin{aligned}
&= \int_G \sum_{k \in L} f(x, k) \rho(x, l) e(k, l) \pi(k + l) dx \\
&= \sum_{k \in L} \int_G f(x, k - l) \rho(x, l) e(k - l, l) dx \pi(k).
\end{aligned}$$

This shows that with respect to the basis $\pi(\ell)$, $\ell \in L$, the operator $\tilde{\alpha}(f)$ is given by a rapidly decreasing (i.e., Schwartz class function) kernel in $\mathcal{S}(L \times L)$. Furthermore, the trace is given by

$$\mathrm{Tr}(\tilde{\alpha}(f)) = \sum_{\ell \in L} \int_G f(x, 0) \rho(x, \ell) dx = \sum_{\ell \in L} \varphi^{\mathcal{B}}(\hat{f}(\ell)),$$

where \hat{f} denotes the Fourier transform on the self-dual group $G = M \times M^\wedge$ w.r.t. the bicharacter ρ .

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